NEHRU COLLEGE OF ENGINEERING AND RESEARCH CENTRE
(NAAC Accredited)
(Approved by AICTE, Affiliated to APJ Abdul Kalam Technological University, Kerala)

DEPARTMENT OF ELECTRONICS & COMMUNICATION ENGINEERING

COURSE MATERIALS

EC 301: DIGITAL SIGNAL PROCESSING

VISION OF THE INSTITUTION

To mould true citizens who are millennium leaders and catalysts of change through excellence in education.

MISSION OF THE INSTITUTION

NCERC is committed to transform itself into a center of excellence in Learning and Research in Engineering and Frontier Technology and to impart quality education to mould technically competent citizens with moral integrity, social commitment and ethical values.

We intend to facilitate our students to assimilate the latest technological know-how and to imbibe discipline, culture and spiritually, and to mould them in to technological giants, dedicated research scientists and intellectual leaders of the country who can spread the beams of light and happiness among the poor and the underprivileged.

ABOUT DEPARTMENT

- Established in: 2002
- Course offered: B.Tech in Electronics and Communication Engineering

  M.Tech in VLSI
- Approved by AICTE New Delhi and Accredited by NAAC
- Affiliated to the University of Dr. A P J Abdul Kalam Technological University.

DEPARTMENT VISION

Providing Universal Communicative Electronics Engineers with corporate and social relevance towards sustainable developments through quality education.

DEPARTMENT MISSION

1) Imparting Quality education by providing excellent teaching, learning environment.

2) Transforming and adopting students in this knowledgeable era, where the electronic gadgets (things) are getting obsolete in short span.

3) To initiate multi-disciplinary activities to students at earliest and apply in their respective fields of interest later.

4) Promoting leading edge Research & Development through collaboration with academia & industry.

PROGRAMME EDUCATIONAL OBJECTIVES

PEOI. To prepare students to excel in postgraduate programmes or to succeed in industry / technical profession through global, rigorous education and prepare the students to practice and innovate recent fields in the specified program/ industry environment.

PEO2. To provide students with a solid foundation in mathematical, Scientific and engineering fundamentals required to solve engineering problems and to have strong practical knowledge required to design and test the system.

PEO3. To train students with good scientific and engineering breadth so as to comprehend, analyze, design, and create novel products and solutions for the real life problems.

PEO4. To provide student with an academic environment aware of excellence, effective communication skills, leadership, multidisciplinary approach, written ethical codes and the life-long learning needed for a successful professional career.

PROGRAM OUTCOMES (POS)

Engineering Graduates will be able to:

1. **Engineering knowledge**: Apply the knowledge of mathematics, science, engineering fundamentals, and an engineering specialization to the solution of complex engineering problems.
2. **Problem analysis**: Identify, formulate, review research literature, and analyze complex engineering problems reaching substantiated conclusions using first principles of mathematics, natural sciences, and engineering sciences.
3. **Design/development of solutions**: Design solutions for complex engineering problems and design system components or processes that meet the specified needs with appropriate consideration for the public health and safety, and the cultural, societal, and environmental considerations.
4. **Conduct investigations of complex problems**: Use research-based knowledge and research methods including design of experiments, analysis and interpretation of data, and synthesis of the information to provide valid conclusions.
5. **Modern tool usage**: Create, select, and apply appropriate techniques, resources, and modern engineering and IT tools including prediction and modeling to complex engineering activities with an understanding of the limitations.
6. **The engineer and society**: Apply reasoning informed by the contextual knowledge to assess societal, health, safety, legal and cultural issues and the consequent responsibilities relevant to the professional engineering practice.
7. **Environment and sustainability**: Understand the impact of the professional engineering solutions in societal and environmental contexts, and demonstrate the knowledge of, and need for sustainable development.
8. **Ethics**: Apply ethical principles and commit to professional ethics and responsibilities and norms of the engineering practice.
9. **Individual and team work**: Function effectively as an individual, and as a member or leader in diverse teams, and in multidisciplinary settings.
10. **Communication**: Communicate effectively on complex engineering activities with the engineering community and with society at large, such as, being able to comprehend and write effective reports and design documentation, make effective presentations, and give and receive clear instructions.
11. **Project management and finance**: Demonstrate knowledge and understanding of the engineering and management principles and apply these to one’s own work, as a member and leader in a team, to manage projects and in multidisciplinary environments.

12. **Life-long learning**: Recognize the need for, and have the preparation and ability to engage in independent and life-long learning in the broadest context of technological change.

PROGRAM SPECIFIC OUTCOMES (PSO)

PSO1: Ability to Formulate and Simulate Innovative Ideas to provide software solutions for Real-time Problems and to investigate for its future scope.

PSO2: Ability to learn and apply various methodologies for facilitating development of high quality
System Software Tools and Efficient Web Design Models with a focus on performance optimization.

PSO3: Ability to inculcate the Knowledge for developing Codes and integrating hardware/software
products in the domains of Big Data Analytics, Web Applications and Mobile Apps to create innovative career path and for the socially relevant issues.

COURSE OUTCOMES
EC 301

SUBJECT CODE: EC 308	
COURSE OUTCOMES	
C301.1	State and prove the fundamental properties and relations relevant to DFT and solve basic problems involving DFT based filtering methods
C301.2	Compute DFT and IDFT using DIT and DIF radix-2 FFT algorithms
C301.3	Design linear phase FIR filters and IIR filters for a given specification
C301.4	Illustrate the various FIR and IIR filter structures for the realization of the given system function
C301.5	Explain the architecture of DSP processor (TMS320C67xx) and the finite word length effects
C301.6	Explain the basic multi-rate DSP operations decimation and interpolation in both time and frequency domains using supported mathematical equations

MAPPING OF COURSE OUTCOMES WITH PROGRAM OUTCOMES

CO'S	PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	PO10	PO11	PO12
C301.1	3	3	3	3	3	3			2	1		
C301.2	3	3	3	3	3	3			2	1		
C301.3	3	3	3	3	3	3			2	1		
C301.4	3	3	3	3	3	3			2	1		
C301.5	3	3	3	3	3	3			2	1		

C301.6	3	3	3	3	3	3			2	1		
C301	3	3	3	3	3	3			2	1		

CO'S	PSO1	PSO2	PSO3
C301.1	3	3	1
C301.2	3	3	1
C301.3	3	3	1
C301.4	3	3	1
C301.5	3	3	1
C301.6	3	3	1
C301	3	3	1

SYLLABUS

COURSE CODE	COURSE NAME	L-T-P-C	YEAR OF INTRODUCTION
EC301	**Digital Signal Processing**	**3-1-0-4**	**2016**

Prerequisite: EC 202 Signals & Systems

Course objectives:

1. To provide an understanding of the principles, algorithms and applications of DSP
2. To study the design techniques for digital filters
3. To give an understanding of Multi-rate Signal Processing and its applications
4. To introduce the architecture of DSP processors

Syllabus

Discrete Fourier Transform and its Properties, Linear Filtering methods based on the DFT, Frequency analysis of signals using the DFT, Computation of DFT, FFT Algorithms, IDFT computation using Radix-2 FFT Algorithms, Efficient computation of DFT of two real sequences and a 2N-Point real sequence, Design of FIR Filters, Design of linear phase FIR Filters using window methods and frequency sampling method, Design of IIR Digital Filters from Analog Filters, IIR Filter Design, Frequency Transformations, FIR Filter Structures, IIR Filter Structures, Introduction to TMS320C67xx digital signal processor, Multi-rate Digital Signal Processing, Finite word length effects in DSP systems, IIR digital filters, FFT algorithms.

Expected outcome:

The students will understand

(i) the principle of digital signal processing and applications.

(ii) the utilization of DSP to electronics engineering

Text Books:

1. Oppenheim A. V., Schafer R. W. and Buck J. R., Discrete Time Signal Processing, 3/e, Prentice Hall, 2007.
2. Proakis J. G. and Manolakis D. G., Digital Signal Processing, 4/e, Pearson Education, 2007.

References:

1. Chassaing, Rulph., DSP applications using C and the TMS320C6x DSK. Vol. 13. John Wiley & Sons, 2003.
2. Ifeachor E.C. and Jervis B. W., Digital Signal Processing: A Practical Approach, 2/e, Pearson Education, 2009.
3. Lyons, Richard G., Understanding Digital Signal Processing, 3/e. Pearson Education India, 2004.
4. Mitra S. K., Digital Signal Processing: A Computer Based Approach, 4/e McGraw Hill (India), 2014.
5. NagoorKani, Digital Signal Processing, 2e, Mc Graw –Hill Education New Delhi, 2013
6. Salivahanan, Digital Signal Processing,3e, Mc Graw –Hill Education New Delhi, 2014 (Smart book)
7. Singh A., Srinivasan S., Digital Signal Processing: Implementation Using DSP Microprocessors, Cenage Learning, 2012.

Course Plan

Module	Course content	Hours	End Sem. Exam Marks
I	The Discrete Fourier Transform: DFT as a linear transformation, Relationship of the DFT to other transforms, IDFT	2	15
	Properties of DFT and examples Circular convolution	4	
	Linear Filtering methods based on the DFT- linear convolution using circular convolution, overlap save and overlap add methods	3	
	Frequency Analysis of Signals using the DFT	2	
II	Computation of DFT: Radix-2 Decimation in Time and Decimation in Frequency FFT Algorithms	3	15
	IDFT computation using Radix-2 FFT Algorithms	2	
	Efficient computation of DFT of Two Real Sequences and a 2N-Point Real Sequence	2	
FIRST INTERNAL EXAM			

III	Design of FIR Filters- Symmetric and Anti-symmetric FIR Filters	2	**15**
	Design of linear phase FIR Filters using Window methods (rectangular, Hamming and Hanning) and frequency sampling Method	6	
	Comparison of Design Methods for Linear Phase FIR Filters	1	
IV	Design of IIR Digital Filters from Analog Filters (Butterworth)	4	**15**
	IIR Filter Design by Impulse Invariance, and Bilinear Transformation	3	
	Frequency Transformations in the Analog and Digital Domain	2	
SECOND INTERNAL EXAM			
V	Block diagram and signal flow graph representations of filters	1	**20**
	FIR Filter Structures: (Linear structures), Direct Form, Cascade Form and Lattice Structure	3	
	IIR Filter Structures: Direct Form, Transposed Form, Cascade Form and Parallel Form	2	
	Computational Complexity of Digital filter structures	1	
	Computer architecture for signal processing : Introduction to TMS320C67xx digital signal processor	2	
VI	Multi-rate Digital Signal Processing: Decimation and Interpolation (Time domain and Frequency Domain Interpretation without proof)	3	**20**
	Finite word length effects in DSP systems: Introduction (analysis not required), fixed-point and floating-point DSP arithmetic, ADC quantization noise	2	

	Finite word length effects in IIR digital filters: coefficient quantization errors	2	
	Finite word length effects in FFT algorithms: Round off errors	2	
END SEMESTER EXAM			

Question Paper Pattern (End Sem Exam)

Maximum Marks: 100 **Time : 3 hours**

The question paper shall consist of three parts. Part A covers modules I and II, Part B covers modules III and IV, and Part C covers modules V and VI. Each part has three questions uniformly covering the two modules and each question can have maximum four subdivisions. In each part, any two questions are to be answered. Mark patterns are as per the syllabus with 40 % for theory and 60% for logical/numerical problems, derivation and proof.

QUESTION BANK

MODULE I				
Q:NO:	**QUESTIONS**	**CO**	**KL**	**PAGE NO:**
1	Derive the relationship of DFT to Fourier transform	CO1	K3	4
2	Explain the following properties of DFT a) Circular Convolution b) Time Reversal	CO1	K2	8
3	Derive the relationship of DFT to Z-transform.	CO1	K3	12
4	Explain the following properties of DFT a) Complex conjugate property b) Circular Convolution	CO1	K2	16
5	Explain the following properties of DFT a) Linearity b) Complex conjugate property	CO1	K2	23
6	Find the circular convolution of $x1\ (n)$ = {1, −1, −2,3, −1},$x2\ (n)$ = {1,2,3} Using i) Concentric circle method ii) Matrix metho	CO1	K3	30
7	Find the output y(n) of a filter whose impulse response is h(n)={1,1,1} and input x(n)={3,-1,0,1,3,2,0,1,2,1} using i) Overlap-save method ii) Overlap-add method	CO1	K3	31
8	Find the output y(n) of a filter whose impulse response is h(n)={1,1,1} and input x(n)={3,-1,0,1,3,2,0,1,2,1} using i) Overlap-save method ii) Overlap-add method K3/CO1 O	CO1	K3	33
9	The first eight points of 14-point DFT of a real valued sequence are {12, - 1+j3, 3+j4, 1-j5, -2+j2, 6+j3, -2-j3, 10,…..} i) Determine the remaining points ii) Evaluate x[0] without computing the IDFT of X(k)? iii) Evaluate IDFT to obtain the real sequence ?	CO1	K3	35
10	Find the remaining samples of the 14-point DFT of the sequence given below X(K)={12,-1+j3,3+j4,1-j5,-2+j2,6+j3,-2-j3,10,…..}	CO1	K3	37
11	Consider the sequence x(n)={1,2,-3,0,1,-1,4,2}. Evaluate the following functions without computing the DFT. i) X(0) ii) X(4) iii) $\sum X(K)$ 7 k=0 iv) $\sum e$ $-j3\pi k$ 4 7 k=0 X(K)	CO1	K3	40

MODULE II				
1	Find the IDFT of the sequence X(k)={10,-2+j2,-2,-2-j2} using DIT algorithm	CO2	K5	49
2	Compute 4-point DFT of a sequence x(n)={0,1,2,3} using DIF algorithm	CO2	K5	51
3	Compute 4-point DFT of a sequence x(n)={0,1,2,3} using DIT algorithm.	CO2	K5	52
4	Compute 4-point DFT of a sequence x(n)={1,0,0,1} using DIF algorithm	CO2	K5	53
5	Compute 4-point DFT of a sequence x(n)={0,1,2,3} using DIF algorithm.	CO2	K5	54
6	Compute 4-point DFT of a sequence x(n)={1,-1,1,-1} using DIT algorithm.	CO2	K5	62
7	Find the 8 point DFT of a real sequence x(n)={1,2,3,4,4,3,2,1} using radix-2 decimation in time algorithm	CO2	K3	65
8	Compute the eight point DFT of the sequence $x(n) = \{ 1\ 0 \le n \le 7\ 0\ otherwise\}$ By using DIF algorithms.	CO2	K3	66
9	Compute the 8 point DFT of x(n) = {2,1,-1,3,5,2,4,1} using radix-2 decimation in time FFT algorithm.	CO2	K3	68
10	Find the 8 point DFT of a real sequence x(n)={1,2,2,2,1,0,0,0,0} using radix-2 decimation in frequency algorithm.	CO2	K3	70
11	Compute the eight point DFT of the sequence $x(n) = \{ 1\ 0 \le n \le 7\ 0\ otherwise\}$ By using DIT algorithm	CO2	K3	72
MODULE III				
1	Design a low pass filter with passband gain of unity, cutoff frequency of 1000Hz and working at a sampling frequency of 5KHz. The length of the	CO3	K6	85

	impulse response should be 7. Use a rectangular window technique.			
2	Design a linear phase FIR low pass filter with cutoff frequency of 2KHz and sampling rate of 8KHz with a filter length 11 using Hanning window	CO3	K6	87
3	Design a FIR filter approximately the ideal frequency response Hd(e jω) = e −jαω for |ω| ≤ π 6 = 0 for π 6 ≤ |ω| ≤π Use Hamming window. Determine the filter coefficients for N=13	CO3	K6	89
4	Explain the frequency sampling method of FIR filter design	CO3	K2	90
5	Give equations for N point Hamming and Hanning window functions. Compare them in terms of main lobe width and side lobe level	CO3	K2	91
6	State the condition for the impulse response of FIR filter to satisfy for constant group and phase delay and for only constant group delay.	CO3	K2	94
7	Differentiate between FIR filters and IIR filters	CO3	K2	95
MODULE IV				
1	Illustrate the design of IIR filters from Analog Filters.	CO4	K3	99
2	Design an analog butterworth filter that has a -2dB passband attenuation at a frequency of 20rad/sec and atleast -10dB stopband attenuation at 30rad/sec.	CO4	K3	102
3	For the given specifications design an analog Butterworth filter. $0.9 \leq \|H(j\Omega)\| \leq 1$ for $0 \leq \Omega \leq 0.2\pi$. $\|H(j\Omega)\| \leq 0.2$ for $0.4\pi \leq \Omega \leq \pi$.	CO4	K3	103
4	Design an analog butterworth filter that has a -2dB passband attenuation at a frequency of 20rad/sec and atleast -10dB stopband attenuation at 30rad/sec.	CO4	K3	105
5	Design an analog butterworth filter that has a -2dB passband attenuation at a frequency of 20rad/sec and atleast -10dB stopband attenuation at 30rad/sec.	CO4	K3	107
6	Illustrate the design of IIR filters from Analog Filters.	CO4	K3	110
7	Design a digital butterworth filter satisfying the constraints 0.707 ≤ |H(e jω)| ≤ 1 for 0 ≤ ω ≤ π 2 |H(e jω)| ≤ 0.2 for 3π 4 ≤ ω ≤ π With T = 1 sec. Use Bilinear transform.	CO4	K6	111
8	Convert the analog filter H(s) given below into a 2 nd	CO4	K3	113

	order butterworth digital filter using impulse invariance technique. H(s) = 1 s 2+√2 s+1			
9	Apply bilinear transformation to H(s)= 2 (s+1)(s+2) with T=1 sec and find H(Z)	CO4	K3	114
10	Design a digital butterworth filter satisfying the constraints 0.707 ≤ |H(e jω)| ≤ 1 for 0 ≤ ω ≤ π 2 |H(e jω)| ≤ 0.2 for 3π 4 ≤ ω ≤ π With T = 1 sec. Use Bilinear transform.	CO4	K6	116
MODULE V				
1	Define a signal flow graph. Draw the signal flow graph of first order digital filter.	CO5	K2	126
2	Sketch a cascade realization of FIR filter structure with complex zeros.	CO5	K3	135
3	Realize the transposed form structure for the system Y(n) = -0.1y(n-1) +0.2y(n-2) +3x(n)+ 3.6x(n-1) +0.6x(n-2)	CO5	K3	138
4	Realize the system with difference equation y(n)= 3 4 y(n − 1) − 1 8 y(n − 2) + x(n) + 1 3 x(n − 1) in cascade form.	CO5	K3	141
5	Draw the direct form I and direct form II structures for the difference equation y(n) = x(n)+0.5x(n-1) +3y(n-1)-2y(n-2)	CO5	K3	143
6	Draw the cascade form structure for a discrete time sequence described H(Z)= 1+ 1 2 z −1 1− 3 4 z−1+ 1 8 z−2	CO5	K3	146
7	Realize the system function using minimum number of multipliers H(Z) = (1+z −1)(1+0.5z −1 + 0.5z −2 + z −3)	CO5	K3	147
8	Obtain the parallel form structure for the system given by the difference equation y(n)= -0.1y(n-1) +0.72y(n-2) +0.7x(n)- 0.252x(n-2)	CO5	K3	149
9	Draw the block diagram of TMS320C67xx and briefly explain function of all blocks.	CO5	K3	151
MODULE VI				
1	Draw the block diagram of ADC quantization noise and explain in detail.	CO6	K3	153
2	Explain the effects of coefficient quantization in FIR and IIR filters.	CO6	K2	154

3	Derive the variance of quantization noise in ADC. Assume step size is Δ.	CO6	K3	156
4	Let $x(n)$ = 0.5 □$u(n)$. Obtain the signals for decimation by 3, interpolation by 3.	CO6	K2	157
5	Write notes on finite word length effects in DSP systems.	CO6	K2	158
6	Let a signal $x(n)$ = 0.5 □$u(n)$ is decimated by 2. What happens to its spectrum?	CO6	K2	162
7	Derive Decimation In Time (DIT) FFT algorithm for 8 point DFT and draw the signal flow graph.	CO6	K5	170
8	Explain the effect in the spectrum of a signal x(n) when it is (i) Decimated by a factor 3 (ii) Interpolated by a factor 2 (5)	CO6	K3	171

APPENDIX 1

CONTENT BEYOND THE SYLLABUS

S:NO;	TOPIC	PAGE NO:
1	Array Signal Processing	180

5/8/19 Friday Monday

MODULE - 1

Discrete Fourier Transform. (DFT)

A discrete time signal $x(n)$ will have its discrete fourier transform as given by following equations:

DFT $\rightarrow X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi nk}{N}}$

IDFT $\rightarrow x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(K)\, e^{j\frac{2\pi nk}{N}}$

Q. 1. Find the 4-point DFT of $x(n) = \{0, 1, 2, 3\}$.

soln :

$N = 4$.

$$X(K) = \sum_{n=0}^{3} x(n)\, e^{-j\frac{2\pi nK}{4}}$$

since $N = 4$, we have to find $X(0), X(1), X(2), X(3)$.

$$X(0) = \sum_{n=0}^{3} x(n)\, e^{0} = x(0) + x(1) + x(2) + x(3)$$

$$= 0 + 1 + 2 + 3$$

$$= \underline{6}$$

$$X(1) = \sum_{n=0}^{3} x(n)\, e^{-j\frac{2\pi n \times 1}{4}}$$

$$= \sum_{n=0}^{3} x(n) \left[\cos \frac{\pi}{2}n - j \sin \frac{\pi n}{2}\right]$$

$$= x(0) \cdot [\cos 0 - j \sin 0] + x(1) \cdot \left[\cos \frac{\pi}{2} - j \sin \frac{\pi}{2}\right] + x(2)\left[\cos \pi - j \sin \pi\right] + x(3)\left[\cos \frac{3\pi}{2} - j \sin \frac{3\pi}{2}\right]$$

$$= 0 + (1 \times -j) + (2 \times -1) + 3 \times (-j \times -1)$$

$$= -j - 2 + 3j$$

$$= \underline{-2 + 2j}$$

$$X(2) = \sum_{n=0}^{3} x(n)\, e^{-j\frac{2\pi n \times 2}{4}} = \sum_{n=0}^{3} x(n)\left(\cos n\pi - j\sin n\pi\right)$$

$$= x(0)\left(\cos 0 - j\sin 0\right) + x(1)\left(\cos \pi - j\sin \pi\right) + x(2)\left(\cos 2\pi - j\sin 2\pi\right) + x(3)\left(\cos 3\pi - j\sin 3\pi\right)$$

$$= 0 + (1\times -1) + (2\times 1) + (3\times -1)$$

$$= -1 + 2 - 3$$

$$= \underline{\underline{-2}}$$

$$X(3) = \sum_{n=0}^{3} x(n)\, e^{-j\frac{2\pi n \times 3}{4}}$$

$$= \sum_{n=0}^{3} x(n)\left(\cos \frac{3n\pi}{2} - j\sin \frac{3n\pi}{2}\right)$$

$$= x(0)\left(\cos 0 - j\sin 0\right) + x(1)\left(\cos \frac{3\pi}{2} - j\sin \frac{3\pi}{2}\right) + x(2)\left(\cos 3\pi - j\sin 3\pi\right) + x(3)\left(\cos \frac{9\pi}{2} - j\sin \frac{9\pi}{2}\right)$$

$$= 0 + (1\times -j \times -1) + 2(-1) + (3\times -j \times 1)$$

$$= j - 2 - 3j$$

$$= \underline{\underline{-2-2j}}$$

$$\therefore X(K) = \{6, -2+2j, -2, -2-2j\}$$

& DF

DFT as a Linear Transformation.

* The formulas for DFT and IDFT can be expressed as:

$$X(K) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

$$x(n) = \frac{1}{N}\sum_{K=0}^{N-1} X(K)\, W_N^{-kn}$$

where, Twiddle factor, $W_N = e^{-j\frac{2\pi}{N}}$

$$x_N = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} \qquad X_N = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}$$

$$W_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

Then,

DFT $\rightarrow X_N = W_N x_N$

IDFT $\rightarrow x_N = \frac{1}{N} W_N^{*} \cdot X_N$

Q.1. Find the DFT of $x(n) = \{0, 1, 2, 3\}$.

soln: $x_N = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 & W_4^6 \\ 1 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix}$$

$$W_N = e^{-j\frac{2\pi}{N}}$$

$$W_4^1 = e^{-j\frac{2\pi}{4}} = e^{-j\pi/2} = (\cos \pi/2 - j \sin \pi/2)$$

$$= -j.$$

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$X_4 = W_4 \, x_N$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0+1+2+3 \\ 0-j-2+3j \\ 0-1+2-3 \\ 0+j-2-3j \end{bmatrix}$$

$$X_4 = \begin{bmatrix} 6 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}.$$

2. Find IDFT of $X(k) = \{6, -2+2j, -2, -2-2j\}$.

soln: $x_N = \frac{1}{N} W_N^* X_N$

$* \rightarrow$ take conjugate of complex terms in W_N^*

$$W_N^* = W_4^* = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

$$x_N = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 6 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 6 - 2+2j - 2 - 2-2j \\ 6 - 2j + -2 + 2 + 2j - 2 \\ 6 + 2 - 2j - 2 + 2 + 2j \\ 6 + 2j + 2 + 2 - 2j + 2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 0 \\ 4 \\ 8 \\ 12 \end{bmatrix}$$

$$x_N = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

6/8/19 Tuesday

* Relationship of DFT to other transforms

* Relationship to Fourier transform.

Fourier transform is given by :

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x(n) e^{-j\omega n}$$

DFT is gn by : $X(K) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi nk}{N}}$

→ comparing the above 2 equ., we can say that

$$X(K) = X(e^{j\omega}) \Big|_{\omega = \frac{2\pi K}{N}}$$

So DFT of $x(n)$ is a sampled version of the Fourier transform of the signals.

* Relationship to Z-Transform

z - transform of sequence is gn by : $X(z) = \sum_{n=0}^{N-1} x(n) z^{-n}$

From the equ. of IDFT, $x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) e^{\frac{j2\pi K}{N} n}$

Substitute $x(n)$ in 1st equ,

$$X(z) = \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{K=0}^{N-1} X(K) e^{\frac{j2\pi K}{N} n} \right] z^{-n}$$

$$= \frac{1}{N} \sum_{K=0}^{N-1} X(K) \left[\sum_{n=0}^{N-1} \left(e^{\frac{j2\pi K}{N}} . z^{-1} \right)^n \right]$$

$$\boxed{\sum_{n=0}^{N-1} c^n = \frac{1}{1-c}}$$

$$X(z) = \frac{1 - z^{-N}}{N} \sum_{K=0}^{N-1} \frac{X(K)}{1 - e^{j\frac{2\pi K}{N}} . z^{-1}}$$

* Properties.

1. Periodicity.

If $X(K)$ is the DFT of $x(n)$, then

→ $x(n+N) = x(n)$

& , $X(K+N) = X(K)$

2. Linearity

If $X_1(K)$ is the DFT of $x_1(n)$ & $X_2(K)$ is the DFT of $x_2(n)$, then

→ $DFT[a_1 x_1(n) + a_2 x_2(n)] = a_1 X_1(K) + a_2 X_2(K)$

where, $X_1(K) = \sum_{n=0}^{N-1} x_1(n) . e^{-j\frac{2\pi nk}{N}}$

$X_2(K) = \sum_{n=0}^{N-1} x_2(n) . e^{-j\frac{2\pi nk}{N}}$

proof

$$\Rightarrow DFT[a_1 x_1(n) + a_2 x_2(n)] = \sum_{n=0}^{N-1} (a_1 x_1(n) + a_2 (x_2(n))) e^{-j\frac{2\pi nk}{N}}$$

$$= a_1 \sum_{n=0}^{N-1} x_1(n) e^{-j\frac{2\pi nk}{N}} + a_2 \sum_{n=0}^{N-1} x_2(n) e^{-j\frac{2\pi nk}{N}}$$

$$= a_1 X_1(K) + a_2 X_2(K)$$

3. Time reversal of signal

If DFT of $x(n)$ is $X(K)$, then,

→ $DFT[x((-n))_N] = DFT[x(N-n)] = X(N-K) = X((-K))_N$.

proof :

$$DFT[x(N-n)] = \sum_{n=0}^{N-1} x(N-n) e^{-j\frac{2\pi nk}{N}}$$

Put $N-n = m \Rightarrow n = N-m$.

$e^{\pm j2\pi K} = 1$.

i.e., $$DFT[x(N-n)] = \sum_{m=0}^{N-1} x(m) e^{-j\frac{2\pi K(N-m)}{N}}$$

$$= \sum_{m=0}^{N-1} x(m) . e^{-j\frac{2\pi KN}{N}} . e^{j\frac{2\pi mk}{N}}$$

$$= \sum_{m=0}^{N-1} x(m) e^{j\frac{2\pi mk}{N}}$$

$$= \sum_{m=0}^{N-1} x(m) . e^{j\frac{2\pi m k}{N}} . e^{-j\frac{2\pi mN}{N}}$$

$$= \sum_{m=0}^{N-1} x(m) . e^{-j\frac{2\pi m}{N}(N-K)} = X(N-K)$$

4. Circular frequency shift

If X(K) is DFT of $x(n)$, then,

$\rightarrow DFT\left[x(n)\, e^{\frac{j2\pi l n}{N}}\right] = X((K-l))_N = X(N+K-l)$

Proof

$$DFT\left[x(n)e^{\frac{j2\pi nl}{N}}\right] = \sum_{n=0}^{N-1} x(n)\, e^{\frac{j2\pi nl}{N}} \cdot e^{-\frac{j2\pi nK}{N}}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi n}{N}(K-l)}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi n}{N}(K-l)} \cdot e^{\frac{j2\pi nN}{N}}$$

$$= \sum_{n=0}^{N-1} x(n) \cdot e^{-j\frac{2\pi n}{N}(N+K-l)}$$

$$= X(N+K-l)$$

5. Complex conjugate property

If DFT of $x(n)$ is $X(K)$, then,

$\rightarrow DFT\left[x^*(n)\right] = X^*(N-K) = X^*((-K))_N$

Proof

$$DFT\left[x^*(n)\right] = \sum_{n=0}^{N-1} x^*(n)\, e^{-\frac{j2\pi nK}{N}} = \sum_{n=0}^{N-1}\left[x(n) \cdot e^{\frac{j2\pi nK}{N}}\right]^*$$

$$= \sum_{n=0}^{N-1}\left[x(n) \cdot e^{\frac{j2\pi nK}{N}} \cdot e^{-\frac{j2\pi nN}{N}}\right]^*$$

$$= \sum_{n=0}^{N-1}\left[x(n) \cdot e^{-j\frac{2\pi n}{N}(N-K)}\right]^*$$

$$= X^*(N-K)$$

6. Circular convolution

Let $x_1(n)$ & $x_2(n)$ are finite duration sequences of length N with DFTs $X_1(K)$ & $X_2(K)$ respectively. Then,

$\rightarrow DFT\left[x_1(n) \circledN x_2(n)\right] = X_1(K) \cdot X_2(K)$

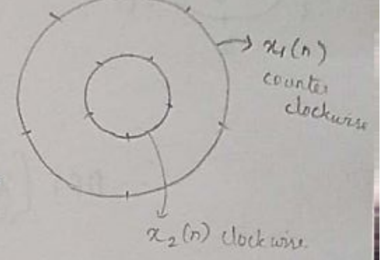
Circular convolution :

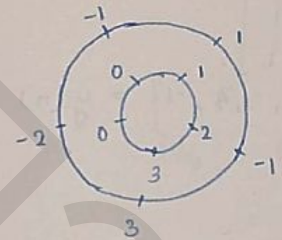
1. Concentric circle method.

Q.1. Find the circular convolution of $x_1(n) = \{1, -1, -2, 3, -1\}$
$x_2(n) = \{1, 2, 3\}$

soln: step 1 :- length of both sequences must be same for performing circular convolution. So we perform zero padding in $x_2(n)$.

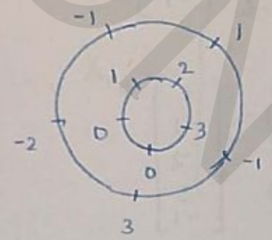
$x_2(n) = \{1, 2, 3, 0, 0\}$



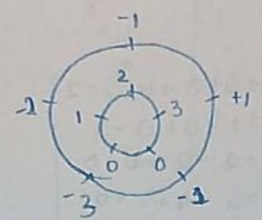


$y(0) = 1\times1 + 2\times-1 + 3\times3 + 0\times-2 + 0\times-1$
$= 1 - 2 + 9 = 8$

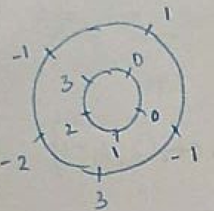
step 2 : rotate inner circle counter clockwise.



$y(1) = 1\times-1 + 2\times1 + 3\times-1 + 0 + 0$
$= -1 + 2 - 3$
$= -2$



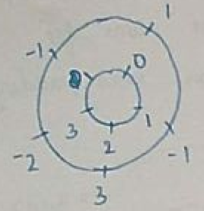
$y(2) = 1\times-2 + 2\times-1 + 3\times1 + 0 + 0$
$= -2 - 2 + 3 = -1$



$y(3) = 1\times3 + 2\times-2 + 3\times-1 + 0 + 0$

$= 3 - 4 - 3$

$= -4$



$y(4) = 1\times-1 + 2\times3 + 3\times-2 + 0 + 0$

$= -1 + 6 - 6$

$= -1$

$\therefore DFT\left[x_1(n) \, Ⓝ \, x_2(n)\right] = \{8, -2, -1, -4, -1\} = y(n)$

2. <u>Matrix method</u>

Q. $x_1(n) = \{1, -1, -2, 3, -1\}$ & $x_2(n) = \{1, 2, 3\}$

soln:

$$x_2(n) = \begin{bmatrix} 1 & 0 & 0 & 3 & 2 \\ 2 & 1 & 0 & 0 & 3 \\ 3 & 2 & 1 & 0 & 0 \\ 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 3 & 2 & 1 \end{bmatrix} \qquad x_1(n) = \begin{bmatrix} 1 \\ -1 \\ -2 \\ 3 \\ -1 \end{bmatrix}$$

$$y(n) = \begin{bmatrix} 1 & 0 & 0 & 3 & 2 \\ 2 & 1 & 0 & 0 & 3 \\ 3 & 2 & 1 & 0 & 0 \\ 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1+0+0+9-2 \\ 2-1+0+0-3 \\ 3-2-2+0+0 \\ 0-3-4+3+0 \\ 0\ \ 0-6+6-1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 \\ -2 \\ -1 \\ -4 \\ -1 \end{bmatrix}$$

3. <u>DFT - IDFT method.</u>

- Given $x_1(n)$ & $x_2(n)$
- Find $X_1(K)$ & $X_2(K)$
- Multiply $X_1(K)$ with $X_2(K)$
- Find IDFT of result $\rightarrow x_1(n) \, Ⓝ \, x_2(n)$.

$\rightarrow DFT\left[x_1(n) \, Ⓝ \, x_2(n)\right] = X_1(K) \cdot X_2(K)$

$\left[x_1(n) \, Ⓝ \, x_2(n)\right] = IDFT\left[X_1(K) \cdot X_2(K)\right]$

Q. $x_1(n) = \{1, 1, 2, 1\}$; $x_2(n) = \{1, 2, 3, 4\}$

soln: ~~$X_1(K)$ =~~

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 & W_4^6 \\ 1 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix}$$

$W_4^1 = e^{-j\frac{2\pi}{N}} = e^{-j\frac{2\pi}{4}} = e^{-j\pi/2}$

$= -j$

$X_N = W_N \cdot x_N$

$$X_{1N} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+1+2+1 \\ 1-j-2+j \\ 1-1+2-1 \\ 1+j-2-j \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$X_{2N} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1+2+3+4 \\ 1-2j-3+4j \\ 1-2+3-4 \\ 1+2j-3-4j \end{bmatrix} = \begin{bmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

$X_1(K) = \{5, -1, 1, -1\}$ $\qquad X_2(K) = \{10, -2+2j, -2, -2-2j\}$

$X_1(K) \cdot X_2(K) = \{50, 2-2j, -2, 2+2j\}$

$$\text{IDFT}\left[X_1(K)\cdot X_2(K)\right] = \frac{1}{N} W_N^{*} \cdot X_N$$

$$x_N = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & +j \end{bmatrix}\begin{bmatrix} 50 \\ 2-2j \\ -2 \\ 2+2j \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 50+2-2j-2+2+2j \\ 50+2j+2+2-2j+2 \\ 50-2+2j+2-2-2j \\ 50-2j-2+2+2j-2 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 52 \\ 56 \\ 44 \\ 48 \end{bmatrix}$$

$$x_N = \begin{bmatrix} 13 \\ 14 \\ 11 \\ 12 \end{bmatrix} \Rightarrow y(n) = \{13, 14, 11, 12\}$$

7/8/19 Wednesday

Linear convolution using Circular convolution.

Q. 1. Find the linear convolution of $x(n) = \{1, 2, 3, 1\}$ and $h(n) = \{1, 1, 1\}$ using circular convolution.

soln: length of $x(n) = 4 = L$.

length of $h(n) = 3 = M$

→ Find $L+M-1 = 4+3-1 = 6$

Making both $x(n)$ & $h(n)$ sequences with length $L+M-1$ and then performing circular convolution will give linear convolution.

$x(n) = \{1, 2, 3, 1, 0, 0\}$

$h(n) = \{1, 1, 1, 0, 0, 0\}$

$$y(n) = h(n) \cdot x(n)$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0+0+0+0 \\ 1+2 \\ 1+2+3 \\ 2+3+1 \\ 3+1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 6 \\ 6 \\ 4 \\ 1 \end{bmatrix}$$

$$y(n) = \{1, 3, 6, 6, 4, 1\}$$

* Filtering of long duration sequences

Suppose an i/p sequence $x(n)$ of very long duration is to be processed with a system having impulse response of finite duration it would cause practical difficulty in convolving the 2 sequences. So, the i/p sequence must be divided into blocks & each block is treated separately. There are 2 methods used for this.

1 : overlap - save method

2 : overlap - add method.

Q.1. Find the o/p $y(n)$ of a filter whose impulse response is $h(n) = \{1, 1, 1\}$ and i/p $x(n) = \{3, -1, 0, 1, 3, 2, 0, 1, 2, 1\}$ using (i) overlap - save method

(ii) overlap - add method.

soln : (i) overlap - save method.

length of $h(n)$, $M = 3$ $\quad x(n) = \{3,-1,0,1,3,2,0,1,2,1\}$

$M-1 = 2$

$\because M-1=2$

$x_1(n) = \{0,0,3,-1,0\}$

$x_2(n) = \{-1,0,1,3,2\}$

$x_3(n) = \{3,2,0,1,2\}$

$x_4(n) = \{1,2,1,0,0\}$

→ find circular convolution with $h(n)$. ; $h(n) = \{1,1,1,0,0\}$

$$y_1(n) = \begin{bmatrix}1&0&0&1&1\\1&1&0&0&1\\1&1&1&0&0\\0&1&1&0&0\\0&0&1&1&1\end{bmatrix}\begin{bmatrix}0\\0\\3\\-1\\0\end{bmatrix} = \begin{bmatrix}-1\\0\\3\\3-1\\3-1\end{bmatrix} = \begin{bmatrix}-1\\0\\3\\2\\2\end{bmatrix} = \{-1,0,3,2,2\}$$

$(M-1)$ discard

$$y_2(n) = \begin{bmatrix}1&0&0&1&1\\1&1&0&0&1\\1&1&1&0&0\\0&1&1&1&0\\0&0&1&1&1\end{bmatrix}\begin{bmatrix}-1\\0\\1\\3\\2\end{bmatrix} = \begin{bmatrix}-1+3+2\\-1+2\\-1+1\\1+3\\1+3+2\end{bmatrix} = \begin{bmatrix}4\\1\\0\\4\\6\end{bmatrix} = \{4,1,0,4,6\}$$

$$y_3(n) = \begin{bmatrix}1&0&0&1&1\\1&1&0&0&1\\1&1&1&0&0\\0&1&1&1&0\\0&0&1&1&1\end{bmatrix}\begin{bmatrix}3\\2\\0\\1\\2\end{bmatrix} = \begin{bmatrix}3+1+2\\3+2+2\\3+2\\2+1\\1+2\end{bmatrix} = \begin{bmatrix}6\\7\\5\\3\\3\end{bmatrix} = \{6,7,5,3,3\}$$

$$y_4(n) = \begin{bmatrix}1&0&0&1&1\\1&1&0&0&1\\1&1&1&0&0\\0&1&1&1&0\\0&0&1&1&1\end{bmatrix}\begin{bmatrix}1\\2\\1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\1+2\\1+2+1\\2+1\\1\end{bmatrix} = \begin{bmatrix}1\\3\\4\\3\\1\end{bmatrix} = \{1,3,4,3,1\}$$

$y(n) = \{3,2,2,0,4,6,5,3,3,4,3,1\}$

(ii) overlap - add method

$x_1(n) = \{3,-1,0,0,0\}$ $\quad x_2(n) = \{1,3,2,0,0\}$

$x_3(n) = \{0,1,2,0,0\}$ $\quad x_4(n) = \{1,0,0,0,0\}$

$M-1$

circular convolution with $h(n) = \{1,1,1,0,0\}$

$$y_1(n) = \begin{bmatrix}1&0&0&1&1\\1&1&0&0&1\\1&1&1&0&0\\0&1&1&1&0\\0&0&1&1&1\end{bmatrix}\begin{bmatrix}3\\-1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}3\\3-1\\3-1\\-1\\0\end{bmatrix} = \begin{bmatrix}3\\2\\2\\-1\\0\end{bmatrix} = \{3,2,2,-1,0\}$$

$$y_2(n) = \begin{bmatrix}1&0&0&1&1\\1&1&0&0&1\\1&1&1&0&0\\0&1&1&1&0\\0&0&1&1&1\end{bmatrix}\begin{bmatrix}1\\3\\2\\0\\0\end{bmatrix} = \begin{bmatrix}1\\1+3\\1+3+2\\3+2\\2\end{bmatrix} = \begin{bmatrix}1\\4\\6\\5\\2\end{bmatrix} = \{1,4,6,5,2\}$$

$$y_3(n) = \begin{bmatrix}1&0&0&1&1\\1&1&0&0&1\\1&1&1&0&0\\0&1&1&1&0\\0&0&1&1&1\end{bmatrix}\begin{bmatrix}0\\1\\2\\0\\0\end{bmatrix} = \begin{bmatrix}0\\1\\1+2\\1+2\\2\end{bmatrix} = \begin{bmatrix}0\\1\\3\\3\\2\end{bmatrix} = \{0,1,3,3,2\}$$

$$y_4(n) = \begin{bmatrix}1&0&0&1&1\\1&1&0&0&1\\1&1&1&0&0\\0&1&1&1&0\\0&0&1&1&1\end{bmatrix}\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix} = \begin{bmatrix}1\\1\\1\\0\\0\end{bmatrix} = \{1,1,1,0,0\}$$

```
  3 2 2 -1 0
+       1 4 6 5 2
              0 1 3 3 2
                    1 1 1 0 0
  3 2 2 0 4 6 5 3 3 4 3 1 0 0 → (M-1) discard
```

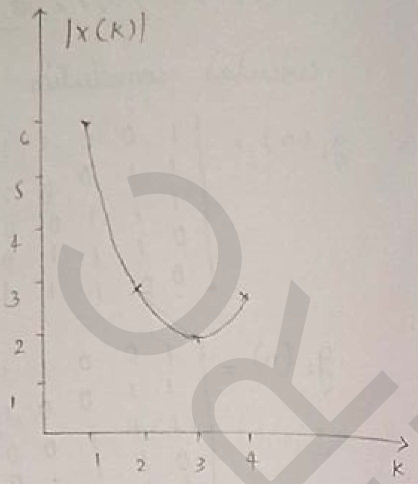
$y(n) = \{3,2,2,0,4,6,5,3,3,4,3,1\}$

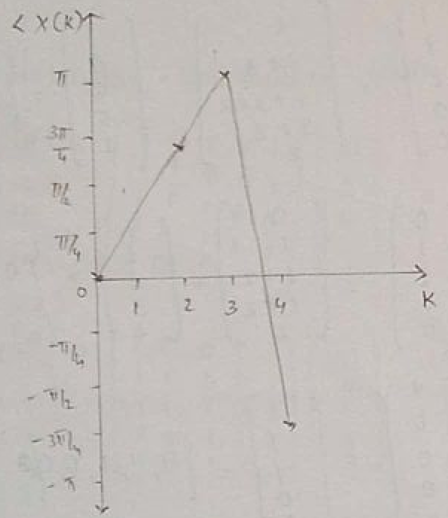
* UQs

Q1. Find the DFT of $x(n) = \{0,1,2,3\}$ and plot $|X(K)|$ & $\angle X(K)$

soln: DFT $[x(n)] = X(K) = \{6, -2+2j, -2, -2-2j\}$

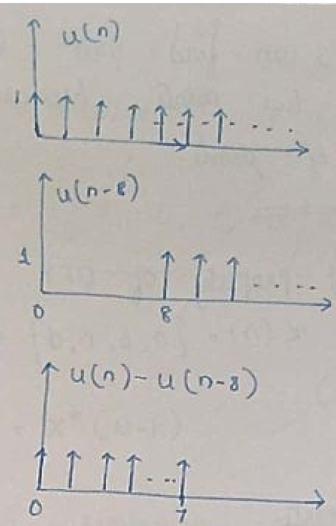
	$\lvert X(K)\rvert$	$\angle X(K)$
6	6	0
-2+2j	2.82	135
-2	2	180
-2-2j	2.82	-135





2. Find the z-transform of $x(n) = u(n) - u(n-8)$ and sample it at 6-points on the unit circle using the relation $X(K) = X(z)\big|_{z=e^{\frac{j2\pi k}{6}}}$; $k = 0,1,\dots 5$. Find the inverse DFT of $X(K)$ & compare it with $x(n)$ & give your comments.

soln:



$\therefore x(n) = \{1,1,1,1,1,1,1,1\}$

$\sum x(n) z^{-n} = X(z) = 1 + z^{-1} + z^{-2} + z^{-3} + \dots z^{-7}$

$X(K) = X(z)\big|_{z=e^{\frac{j2\pi k}{6}}} = X(z)\big|_{z=e^{\frac{j\pi k}{3}}}$

$= 1 + e^{\frac{-jk\pi}{3}} + e^{-j\frac{2\pi}{3}k} + e^{-j\frac{3\pi k}{3}} + e^{-j\frac{4\pi k}{3}} + e^{-j\frac{5\pi k}{3}} + e^{-j\frac{6\pi k}{3}} + e^{-j\frac{7\pi k}{3}}$

$= 1 + e^{-j\frac{\pi k}{3}} + e^{-j\frac{2\pi k}{3}} + 1 + e^{-j\frac{4\pi k}{3}} + e^{-j\frac{5\pi k}{3}} + 1 + e^{-j\frac{\pi k}{3}(6+1)}$

$\because k = 0 \to 5$

$= \left(1 + e^{-j\frac{\pi k}{3}} + e^{-j\frac{2\pi k}{3}} + 1 + e^{-j\frac{4\pi k}{3}} + e^{-j\frac{5\pi k}{3}}\right) + 1 + e^{-j\frac{\pi k}{3}}$

$= 2 + 2e^{-j\frac{\pi k}{3}} + e^{-j\frac{2\pi k}{3}} + 1 + e^{-j\frac{4\pi k}{3}} + e^{-j\frac{5\pi k}{3}}$

$x'(n) = \{2, 2, 1, 1, 1, 1\} \rightarrow$ coefficients

Comparing $x'(n)$ & $x(n)$ we can find that time domain aliasing occurs in the first two points because $X(z)$ is not sampled with sufficient no. of points.

17.8.19 Saturday

3. (i) State circular freq shift property of DFT.

* (ii) 4-point DFT of a signal $x(n) = \{a, b, c, d\}$ is $X(K)$. Find the IDFT of $X(K-2)$.

soln:

(i) Circular freq shift property

$$DFT\left[x(n)\, e^{\frac{j2\pi l n}{N}}\right] = X(N+K-l)$$

(ii)

$$x(n)\, e^{\frac{j2\pi l n}{N}} = IDFT\left[X(N+K-l)\right]$$

$$IDFT\left[X(K+N-l)\right] = IDFT\left[X(4+K-l)\right]$$

Since we have to find IDFT $[X(K-2)]$, put $l = 6$,

$$\Rightarrow IDFT\left[X(4+K-6)\right] = IDFT\left[X(K-2)\right]$$

$$Y(K) = IDFT\left[X(K-2)\right] = x(n).e^{\frac{j2\pi \times 6 \times n}{4}}$$

$$= x(n).e^{j3\pi n}$$

put $n = 0, 1, 2, 3$.

$Y(0) = x(0).e^{0} = a$

$Y(1) = x(1).e^{j3\pi} = b.-1 = \underline{-b}$

$Y(2) = x(2).e^{j6\pi} = c.1 = \underline{c}$

$Y(3) = x(3).e^{j9\pi} = d.-1 = \underline{-d}$

$$Y(K) = \{a, -b, c, -d\}$$

NOTE :

$$X(K) = X^*(N-K)$$

4. Find the remaining samples of the 14-point DFT of a sequence given below.

$$X(K) = \{12, -1+3j, 3+4j, 1-5j, -2+2j, 6+3j, -2-3j, 10, \ldots\}$$

(indices 0, 1, 2, 3, 4, 5, 6, 7, …, 13 written above the samples)

soln:

$\rightarrow X(8) = X^*(14-8) = X^*(6)$

$X(6) = -2-3j \Rightarrow X^*(6) = \underline{-2+3j}$

$\rightarrow X(9) = X^*(14-9) = X^*(5)$

$X(5) = 6+3j \Rightarrow X^*(5) = \underline{6-3j}$

$\rightarrow X(10) = X^*(14-10) = X^*(4) = \underline{-2-2j}$

$\rightarrow X(11) = X^*(14-11) = X^*(3) = \underline{1+5j}$

$\rightarrow X(12) = X^*(14-12) = X^*(2) = \underline{3-4j}$

$\rightarrow X(13) = X^*(14-13) = X^*(1) = \underline{-1-3j}$

$\therefore X(K) = \{12, -1+3j, 3+4j, 1-5j, -2+2j, 6+3j, -2-3j, 10, \underline{-2+3j}, \underline{6-3j}, \underline{-2-2j}, \underline{1+5j}, \underline{3-4j}, \underline{-1-3j}\}$

5. Consider the sequence $x(n) = \{1, 2, -3, 0, 1, -1, 4, 2\}$. Evaluate the following functions without computing the DFT.

(a) $X(0)$ (b) $X(4)$ (c) $\sum_{k=0}^{7} X(k)$ (d) $\sum_{k=0}^{7} e^{-j\frac{3\pi k}{4}} X(k)$

(e) $\sum_{k=0}^{7} |X(k)|^2$

soln:

$N = 8$. $X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi nk}{N}}$

(a) $X(0) = \sum_{n=0}^{7} x(n) = x(0) + x(1) + \dots + x(7) = \underline{6}$

(b) $X(4) = \sum_{n=0}^{7} x(n) \cdot e^{-j\frac{2\pi n \times 4}{8}}$

$= \sum_{n=0}^{7} x(n) \cdot e^{-j\pi n} = \sum_{n=0}^{7} x(n) \cdot \cos n\pi$

$= 1 \cdot \cos 0 + 2 \times \cos \pi - 3 \times \cos 2\pi + 0 + 1 \cdot \cos 4\pi + -1 \cdot \cos 5\pi + 4 \cdot \cos 6\pi + 2 \cdot \cos 7\pi$

$= 1 - 2 - 3 + 1 + 1 + 4 - 2$

$= \underline{0}$

19/8/19 Monday.

(c) $\sum_{k=0}^{7} X(k)$ → since $X(k)$ is present, IDFT is to be used

$x(n) = \frac{1}{N} \sum_{k=0}^{7} X(k)\, e^{\frac{j2\pi nk}{N}}$

$x(n) = \frac{1}{8} \sum_{k=0}^{7} X(k)\, e^{\frac{j2\pi nk}{8}}$

$x(0) = \frac{1}{8} \sum_{k=0}^{7} X(k) \quad \Rightarrow \quad \sum_{k=0}^{7} X(k) = x(0) \cdot 8$

$= 1 \times 8$

$= \underline{8}$

(d) $\sum_{k=0}^{7} e^{-j\frac{3\pi k}{4}} X(k)$ → IDFT want to use.

Property: Circular Shift of a Sequence.

$DFT\left[x((n-m))_N\right] = e^{\frac{-j2\pi km}{N}} X(k)$

$x((n-m))_N = IDFT\left[e^{\frac{-j2\pi km}{N}} X(k)\right]$

$\Rightarrow x((n-m))_N = \frac{1}{N} \sum_{k=0}^{N-1} \left(X(k) \cdot e^{\frac{-j2\pi km}{N}}\right) \cdot e^{\frac{j2\pi nk}{N}}$

Put $n = 0$,

$x((0-m))_N = \frac{1}{8} \sum_{k=0}^{7} X(k) \cdot e^{\frac{-j2\pi km}{8}}$

$= \frac{1}{8} \sum_{k=0}^{7} X(k) \cdot e^{\frac{-j\pi km}{4}}$

Comparing the above with the question, $m = 3$.

$x((0-m))_N = \frac{1}{8} \sum_{k=0}^{7} X(k) \cdot e^{-j\frac{3\pi km}{4}}$

$\sum_{k=0}^{7} X(k) \cdot e^{-j\frac{3\pi k}{4}} = x((0-3))_8 \times 8 \qquad \because x((n-m))_N = x(N+n-m)$

$= x(8+0-3) \times 8$

$= x(5) \cdot 8$

$= -1 \cdot 8 = \underline{-8}$

(e) $\sum_{k=0}^{7} |X(k)|^2$

Parseval's Theorem: $\sum_{k=0}^{N-1} |X(k)|^2 = N \cdot \sum_{n=0}^{N-1} |x(n)|^2$

According to Parseval's theorem,

$\sum_{k=0}^{7} |X(k)|^2 = 8 \sum_{n=0}^{7} |x(n)|^2$

$= 8\left[x(0)^2 + x(1)^2 + x(2)^2 + x(3)^2 + x(4)^2 + x(5)^2 + x(6)^2 + x(7)^2\right]$

$= 8\left[1 + 4 + 9 + 0 + 1 + 1 + 16 + 4\right] = 8 \times 36$

$= \underline{288}$

6. If $x(n) = \{1,2,3,4\}$, find DFT[DFT $x(n)$] without finding the DFT.

soln: $DFT[DFT\ x(n)] = N x((-n))_N$ $\qquad \{1, 2, 3, 4\}$

$N = 4,$

$X(K) = 4x(N-n)$

$\Rightarrow X(K)\big|_{n=0,1,2,3}$ is gn by,

$n=0 \rightarrow 4[x(4-0)] = 4x(4) = 4 \times x(0) = 4$

$n=1 \rightarrow 4[x(4-1)] = 4 \cdot x(3) = 4 \times 4 = 16$

$n=2 \rightarrow 4[x(4-2)] = 4 \cdot x(2) = 4 \times 3 = 12$

$n=3 \rightarrow 4[x(4-3)] = 4 \cdot x(1) = 4 \times 2 = 8$

$\therefore X(K) = \{4, 16, 12, 8\}$

7. State Parseval's property.

DFT of a real value signal $X(K) = \{j, 1+j, A, 1-j, -1, B, -1-j, C\}$

Find the energy of the signal.

soln: Parseval's theorem, $\sum_{k=0}^{N-1} |X(K)|^2 = N \cdot \sum_{n=0}^{N-1} |x(n)|^2$

Inorder to find energy, we have to find A, B, C

$X(K) = X^*(N-K)$ $\qquad N = 8$

$\rightarrow X(2) = X^*(8-2) = X^*(6) = -1+j = A$

$\rightarrow X(5) = X^*(8-5) = X^*(3) = 1+j = B$

$\rightarrow X(7) = X^*(8-7) = X^*(1) = 1-j = C.$

Energy: $\sum_{K=0}^{N-1} |X(K)|^2 = \sum_{k=0}^{7} |X(K)|^2 = |X(0)|^2 + |X(1)|^2 + |X(2)|^2 + |X(3)|^2 + |X(4)|^2 + |X(5)|^2 + |X(6)|^2 + |X(7)|^2$

$= (\sqrt{0+1})^2 + (\sqrt{1+1})^2 + (\sqrt{1+1})^2 + (\sqrt{1+1})^2 + (\sqrt{1})^2 + (\sqrt{1+1})^2 + (\sqrt{1+1})^2 + (\sqrt{1+1})^2$

$= 1 + 2 + 2 + 2 + 1 + 2 + 2 + 2$

$= 14$

Energy $= \sum_{n=0}^{7} |x(n)|^2$

According to Parseval's theorem, $\sum_{k=0}^{7} |X(K)|^2 = N \sum_{k=0}^{7} |x(n)|^2$

$\sum_{k=0}^{7} |x(n)|^2 = \frac{1}{N} \cdot \sum_{k=0}^{7} |X(K)|^2$

$= \frac{14}{8} = 1.75\ J$

20/8/19
Tuesday

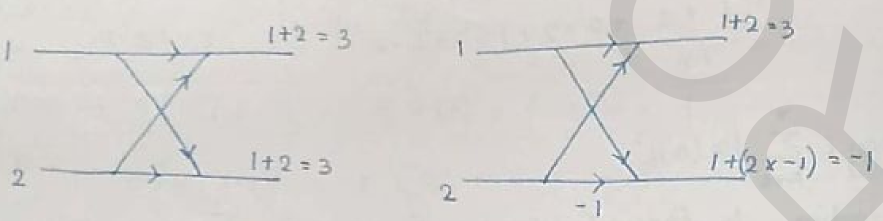
* MODULE - 2

FAST FOURIER TRANSFORM (FFT)

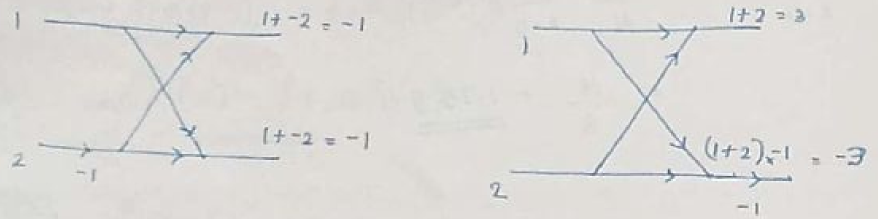
Two types of FFT

1. Radix - 2 Decimation in Time - FFT (DIT-FFT)
2. Radix - 2 Decimation in Freq - FFT (DIF-FFT)

eg:

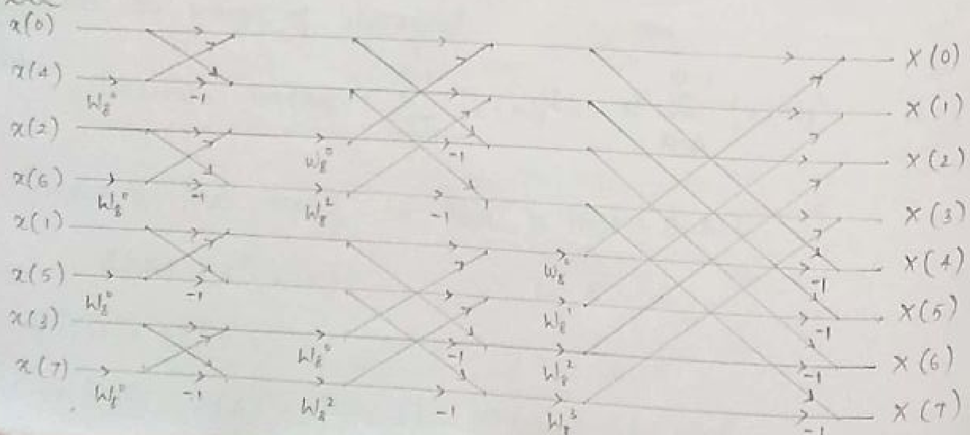


Butterfly diagram.



8 - Point DIT - FFT

Bit reversal



$W_N = e^{\frac{-j2\pi}{N}}$ - Tiddle factor

$W_8^1 = e^{\frac{-j2\pi}{8}} = e^{-j\pi/4} = \cos \pi/4 - j \sin \pi/4$

$\Rightarrow W_8^0 = 1$

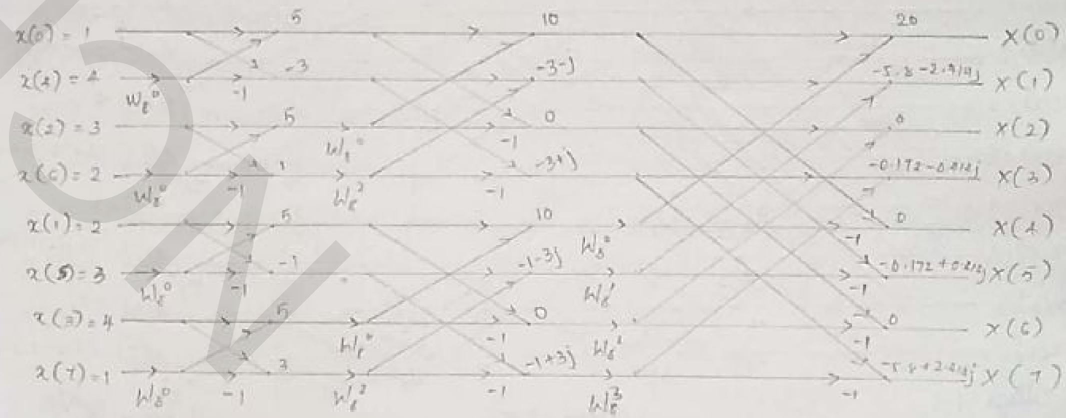
$W_8^1 = 0.707 - j\,0.707$

$W_8^2 = -j$

$W_8^3 = -0.707 - j0.707.$

Q.
1. Find the DIT-FFT of $x(n) = \{1, 2, 3, 4, 4, 3, 2, 1\}$

soln $\Rightarrow X(k) = \{20, -5.8 - 2.414j, 0, -0.172 - 0.414j, 0, -0.172 + 0.414j, 0, -5.8 + 2.414j\}$



stage 1 :

→ 1 + 4 = 5

→ 1 + (-4) = -3

→ 3 + 2 = 5

→ 3 + (-2) = 1

→ 2 + 3 = 5

→ -3 + 2 = -1

→ 4 + 1 = 5

→ 4 + (-1) = 3

stage 2 :

→ 5 + 5 = 10

→ $-3 + (1 \times W_8^2) = -3 + (-j) = -3 - j$

→ 5 + (-5) = 0

→ $-3 - (W_8^2) = -3 + j$

→ 5 + 5 = 10

→ $-1 + 3 \times W_8^2 = -1 - 3j$

→ 5 + (-5) = 0

→ $-1 + -3 W_8^2 = -1 + 3j$

stage 3 :

→ 10 + 10 = 20

→ $(-3-j) + (-1-3j) \cdot W_8^1 = (-3-j) + (-1-3j) \times (0.707 - j0.707)$
$= -5.828 - 2.414j$

→ $0 + 0 W_8^2 = 0$

→ $(-3+j) + (-1+3j) W_8^3 = -0.172 - 0.414j$

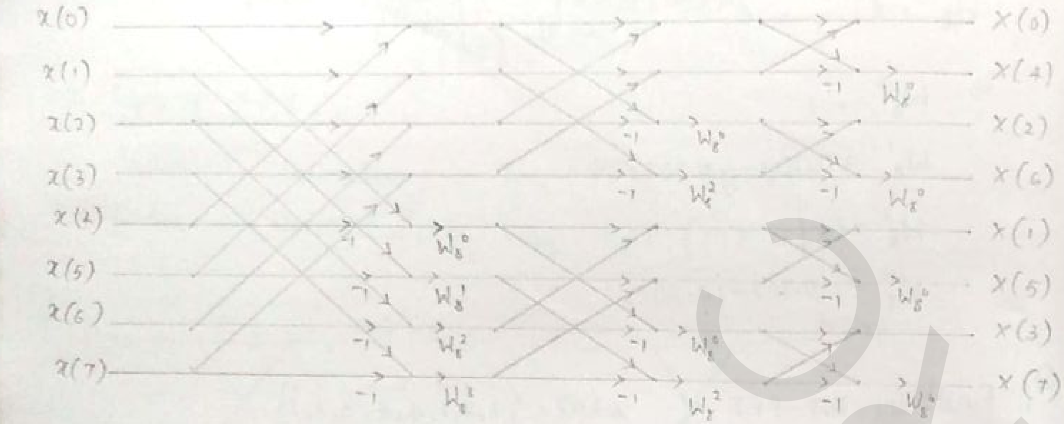
→ 10 - 10 = 0

→ $(-3-j) + (-1-3j) \times -W_8^1 = -0.172 + 0.414j$

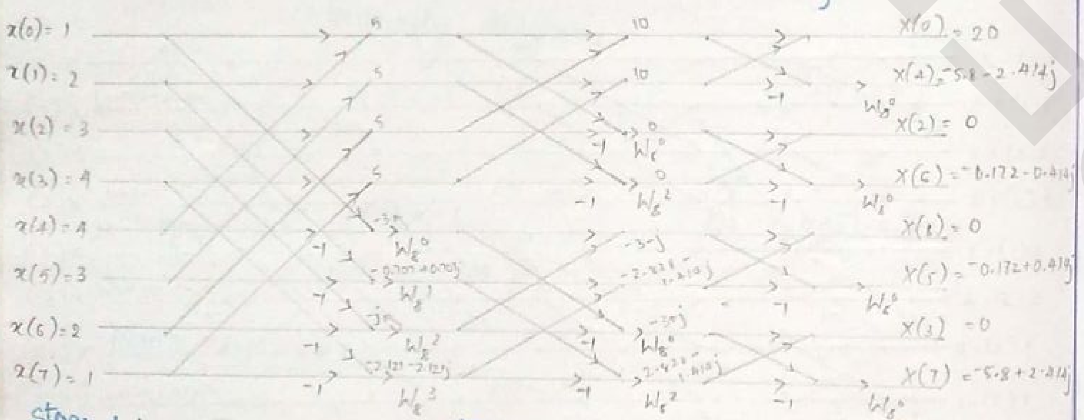
→ $0 + 0 \times -W_8^2 = 0$

→ $(-3+j) + (-1+3j) \times -W_8^3 = -5.8 + 2.414j$

DIF- FFT



Q. Find the DIF-FFT of $x(n) = \{1,2,3,4,4,3,2,1\}$



stage 1 :

→ $1+4=5$
→ $2+3=5$
→ $3+2=5$
→ $4+1=5$
→ $1+(-4) = -3 \times W_8^0 = -3$
→ $2+(-3) = -1 \times W_8^1 = -0.707+0.707j$
→ $3+(-2) = 1 \times W_8^2 = -j$
→ $4+(-1) = 3 \times W_8^3 = -2.121-2.121j$

stage 2 :

→ $5+5 = 10$
→ $5+5 = 10$
→ $5+(-5) = 0 \times W_8^0 = 0$
→ $5+(-5) = 0 \times W_8^2 = 0$
→ $-3+(-j) = -3-j$
→ $(-0.707+0.707j)+(-2.121-2.121j) = -2.828-1.414j$
→ $-3+(-j \times -1) = (-3+j).W_8^0 = -3+j$
→ $-[(0.707+0.707j) \mp (-2.121-2.121j)].W_8^2 = 2.828-1.414j$

stage 3 :

→ $10+10 = 20$
→ $10+(-10) = 0$
→ 0
→ 0
→ $(-3-j)+(-2.828-1.414j) = -5.8-2.414j$
→ $(-3-j)-(-2.8-1.414j) = -0.172+0.414j$
→ $-0.172-0.414j$
→ $-5.8+2.414j$

21/8/19. Wednesday

IDFT Using FFT Algorithm

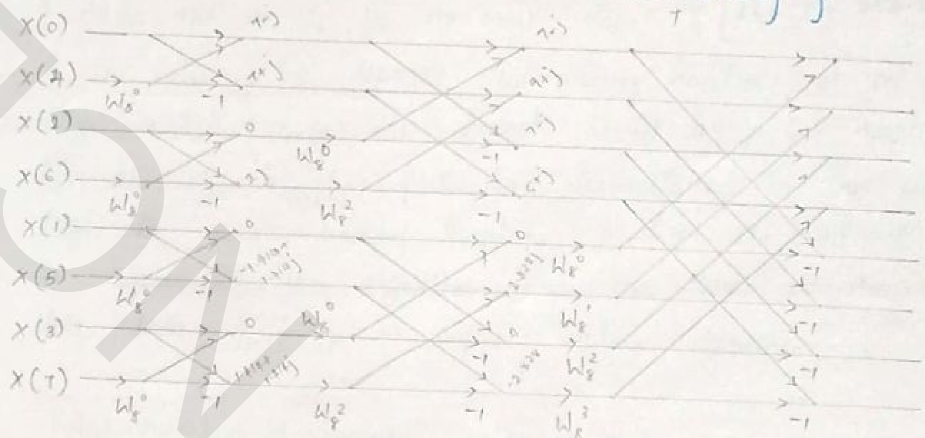
steps :

1. Find the complex conjugate of given sequence $X(K)$
2. Perform DIT or DIF to the complex conjugate obtained in step 1.
3. Divide the final answer obtain by N & find the complex conjugate.

Q1. Compute the IDFT of the sequence

$X(K) = \{7, -0.707-0.707j, -j, 0.707-0.707j, j, 0.707+0.707j, j, -0.707+0.707j\}$

soln:



stage 1 :

→ $7+j$
→ $7-j$
→ $-j+j = 0$
→ $-j-j = -2j$
→ $-0.707-0.707j + (0.707+0.707j) = 0$
→ $(-0.707-0.707j)-(0.707+0.707j) = -1.414-1.414j$

stage 1 :

→ $7+(-j) = 7-j$
→ $7-(-j) = 7+j$
→ $+j-j = 0$
→ $j-(-j) = 2j$
→ $(-0.707+0.707j)+(0.707-0.707j) = 0$
→ $(-0.707+0.707j)-(0.707-0.707j) = -1.414+1.414j$
→ $(0.707+0.707j)+(-0.707-0.707j) = 0$
→ $(0.707+0.707j)-(-0.707-0.707j) = 1.414+1.414j$

Stage 2 :

→ $7-j+0 = 7-j$

→ $7+j+(2j \times -j) = 7+j+2$
$= 9+j$

→ $7-j-0 = 7-j$

→ $7+j-2j \times -j = 5+j$

→ 0

→ $-1.414+1.414j + (+1.414+1.414j) \cdot -j$
$= 0$

→ 0

→ $(-1.414+1.414j) - (+1.414+1.414j) \cdot -j$
$= -2.828 + 2.828j$

Stage 3 :

→ $7-j+0 = 7-j$

→ $9+j + 2.828j(0.707-0.707j) = 10.999+2.99j$

→ $7-j+0 = 7-j$

→ $5+j+(-2.828+2.828j)(-0.707-0.707j) = 6.999+2.99j$

→ $7-j-0 = 7-j$

→ $9+j - 2.828j(0.707-0.707j) = 7-0.99j$

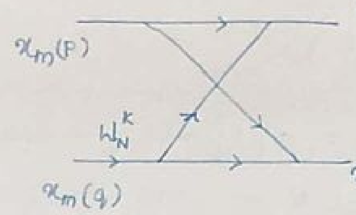
→ $7-j-0 = 7-j$

→ $5+j-(-2.828+2.828j)(-0.707-0.707j) = 3-0.99j$

NOTE :

Inplace computation of DFT.

A basic butterfly to perform FFT algorithm can be shown as



$x_{m+1}(p) = x_m(p) + W_N^k x_m(q)$

$x_{m+1}(q) = x_m(p) - W_N^k x_m(q)$

If we use 'm' to represent the stage & p & q to represent notes in each stage, we can perform the operation as shown above & obtain the results for the next stage.

If the notes p & q represent the memory location, at the first stage $x_m(p)$ & $x_m(q)$ are stored in p & q respectively. After the o/ps $x_{m+1}(p)$ & $x_{m+1}(q)$ are obtained in the second stage the same memory locations p & q are used to store the values. So the algorithm that use same location to store both i/p & o/p is called inplace algorithm.

Total number of complex multiplications

(1) Total no. of complex multiplication using direct evaluation of DFT $= N^2$

(2) Total no. of complex multiplication using FFT algorithm.
$= \frac{N}{2} \log_2 N$

* Speed improvement factor $= \frac{N^2}{\frac{N}{2}\log_2 N}$

* No. of complex additions using FFT $= N \log_2 N$

Q. Find the no. of complex multiplications involved in the calculation of 1024 point DFT using
(a) direct evaluation (b) Radix-2 FFT

soln: (a) by direct evaluation

$= N^2 = 1024^2$

$= \underline{1048576}$

(b) Radix-2 FFT $= \frac{N}{2} \log_2 N$

$= \underline{5120}$

Efficient computation of DFT of 2 real sequences.

Inorder to find DFT of 2 real sequences $x_1(n)$ & $x_2(n)$ both of length N, let us define a complex value sequence $x(n)$ such that $x(n) = x_1(n) + j\,x_2(n)$. Since DFT is linear we can write DFT of $x(n) = X(K) = X_1(K) + j\,X_2(K)$ where $X_1(K)$ & $X_2(K)$ are DFTs of $x_1(n)$ & $x_2(n)$ respectively.
Now we can express $x_1(n)$ & $x_2(n)$ in terms of $x(n)$ as follows:

$$x_1(n) = \frac{x(n) + x^*(n)}{2}$$

$$x_2(n) = \frac{x(n) - x^*(n)}{2j}$$

Hence

$$X_1(K) = \frac{1}{2}\left[\frac{DFT[x(n)] + DFT[x^*(n)]}{2}\right] \quad —①$$

$$X_2(K) = \frac{1}{2j}\left[DFT[x(n)] - DFT[x^*(n)]\right] \quad —②$$

$DFT[x(n)] = X(K)$ & $DFT[x^*(n)] = X^*(N-K)$

$\Rightarrow$ ① & ② becomes

$$X_1(K) = \frac{1}{2}\left[X(K) + X^*(N-K)\right]$$

$$X_2(K) = \frac{1}{2j}\left[X(K) - X^*(N-K)\right]$$

Q. Given $g(n) = \{1, 0, 1, 0\}$ & $h(n) = \{1, 2, 2, 1\}$. Find the 4-point DFTs of these 2 sequences using a single 4-point DFT.

soln: $g(n) = \{1, 0, 1, 0\}$

$h(n) = \{1, 2, 2, 1\}$ $\qquad N = 4$

$x(n) = \{1+j, 2j, 1+2j, j\}$

Using matrix method, DFT, $X_N = W_N x_N$

$$x_4 = \begin{bmatrix} 1+j \\ 2j \\ 1+2j \\ j \end{bmatrix} \qquad W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$X_4 = W_4 x_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}\begin{bmatrix} 1+j \\ 2j \\ 1+2j \\ j \end{bmatrix}$$

$$= \begin{bmatrix} 1+j+2j+1+2j+j \\ 1+j+2-1-2j-1 \\ 1+j-2j+1+2j-j \\ 1+j-2-1-2j+1 \end{bmatrix} = \begin{bmatrix} 2+6j \\ 1-j \\ 2 \\ -1-j \end{bmatrix}$$

$X_k(K) = \{2+6j, 1-j, 2, -1-j\}$

$X^*(N-K) = X^*(4-K)$

Put $K = 0, 1, 2, 3$

$X^*(4-0) = X^*(4) = X^*(0) = 2-6j$

$X^*(4-1) = X^*(3) = -1+j$

$X^*(4-2) = X^*(2) = 2$

$X^*(4-3) = X^*(1) = 1+j$

$X^*(N-K) = \{2-6j, -1+j, 2, 1+j\}$

$\Rightarrow G(K) = \frac{1}{2}\left[X(K) + X^*(N-K)\right]$

$= \frac{1}{2}\left[(2+6j+2-6j), (1-j+(-1+j)), (2+2), (-1-j+1+j)\right]$

$= \frac{1}{2}\{4, 0, 4, 0\}$

$G(K) = \{2, 0, 2, 0\}$

$H(K) = \frac{1}{2j}\left[X(K) - X^*(N-K)\right]$

$= \frac{1}{2j}\left[(2+6j)-(2-6j), 1-j-(-1+j), 2-2, -1-j-(1+j)\right]$

$= \frac{1}{2j}\{12j, 2-2j, 0, -2-2j\}$

$= \{6, -j-1, 0, +j-1\}$

$H(K) = \{6, -1-j, 0, -1+j\}$

26/8/19 Monday

Efficient computation of DFT of a 2N point Real sequence.

Suppose that $g(n)$ is a real value sequence of 2N points. Now we define $x_1(n) = g(2n)$; $x_2(n) = g(2n+1)$. Thus we have sub divided the 2N point sequence to two N-points sequences. Let $x(n)$ be complex value sequences :

$x(n) = x_1(n) + j\, x_2(n)$.

From the results of previous sections :

$X_1(K) = \frac{1}{2}\left[X(K) + X^*(N-K)\right]$

$X_2(K) = \frac{1}{2j}\left[X(K) - X^*(N-K)\right]$

We express the 2N-point DFT in terms of two N-point DFTs $X_1(K)$ & $X_2(K)$. For that we use calculations as in DIT algorithm.

$$G(K) = \sum_{n=0}^{N-1} g(2n) W_{2N}^{2nK} + \sum_{n=0}^{N-1} g(2n+1) W_{2N}^{(2n+1)K}$$

$$= \sum_{n=0}^{N-1} x_1(n) W_N^{nK} + W_{2N}^{K} \sum_{n=0}^{N-1} x_2(n) W_N^{nK}$$

$$\Rightarrow G(K) = X_1(K) + W_{2N}^{K} X_2(K)$$

$$G(K+N) = X_1(K) - W_{2N}^{K} X_2(K)$$

Q1. Find the 8-point DFT using two 4-point DFTs of $x(n) = \{1, 2, 3, 4, 4, 3, 2, 1\}$

soln : $g(2n) = \{1, 3, 4, 2\}$

$g(2n+1) = \{2, 4, 3, 1\}$

$\{1+2j,\ 3+4j,\ 4+3j,\ 2+j\}$

~~$x(k) = \begin{bmatrix}1+2j\\3+4j\\4+3j\\2+j\end{bmatrix} \begin{bmatrix}1&1&1&1\\1&-j&-1&j\\1&-1&1&-1\\1&j&-1&-j\end{bmatrix} = $~~

$$X(k) = \begin{bmatrix}1&1&1&1\\1&-j&-1&j\\1&-1&1&-1\\1&j&-1&-j\end{bmatrix}\begin{bmatrix}1+2j\\3+4j\\4+3j\\2+j\end{bmatrix} = \begin{bmatrix}1+2j+3+4j+4+3j+2+j\\1+2j-3j+4-4-3j+2j-1\\1+2j-3-4j+4+3j-2-j\\1+2j+3j-4-4-3j-2j+1\end{bmatrix}$$

$$= \begin{bmatrix}10+10j\\-2j\\0\\-6\end{bmatrix} = \{10+10j,\ -2j,\ 0,\ -6\}$$

$N = 4$

$X^*(N-k) = X^*(4-k)$ put $k = 0,1,2,3$

$X^*(4-0) = X^*(4) = 10-10j$

$X^*(4-1) = X^*(3) = -6$

$X^*(4-2) = X^*(2) = 0$

$X^*(4-3) = X^*(1) = 2j$

$$X_1(k) = \frac{1}{2}\left[X(k) + X^*(N-k)\right]$$

$$= \frac{1}{2}\{(10+10j+10-10j),\ (-2j+-6),\ (0+0),\ (-6+2j)\}$$

$$= \frac{1}{2}\{20,\ -6-2j,\ 0,\ -6+2j\}$$

$$= \{10,\ -3-j,\ 0,\ -3+j\}$$

$$X_2(k) = \frac{1}{2j}\left[X(k) - X^*(N-k)\right]$$

$$= \frac{1}{2j}\{10+10j-(10-10j),\ -2j+6,\ 0-0,\ -6-2j\}$$

$$= \frac{1}{2j}\{20j,\ 6-2j,\ 0,\ -6-2j\}$$

$$= \{10,\ -1-3j,\ 0,\ -1+3j\}$$

$2N = 8$

$G(k) = X_1(k) + W_8^k X_2(k)$

$G(k+N) = X_1(k) - W_8^k X_2(k)$

$W_8^k = e^{-j\frac{2\pi}{8}}$

$\Rightarrow G(0) = X_1(0) + W_8^0 X_2(0)$

$= 10 + 1\times 10 = 20$

$G(1) = X_1(1) + W_8^1 X_2(1)$

$= (-3-j) + (0.707 - j0.707)(-1-3j)$

$G(1) = -5.828 - 2.414j$

$G(2) = X_1(2) + W_8^2 X_2(2)$

$= 0 + -j\cdot 0 = 0$

$G(3) = X_1(3) + W_8^3(X_2(3))$

$= -3+j + (-0.707-0.707j)(-1+3j)$

$= -0.172 - 0.414j$

$G(0+4) = G(4) = X_1(0) - W_8^0 X_2(0) = 0$

$G(1+4) = G(5) = X_1(1) - W_8^1 X_2(1) = -0.172 + 0.414j$

$G(2+4) = G(6) = X_1(2) - W_8^2 X_2(1) = 0$

G, G

$G(3+4) = G(7) = X_1(3) - W_8^3 X_2(3)$

$= -5.828 + 2.414j$

~~$G(k) = \{20, -5.828 - 2.414j, 0, -0.172 - 0.414j\}$~~

~~$G(k+N) = \{0, -0.172 + 0.414j, 0, -5.828 + 2.414j\}$~~

$G(k) = \{20, -5.828 - 2.414j, 0, -0.172 - 0.414j, 0,$
$-0.172 + 0.414j, 0, -5.828 + 2.414j\}$

27/8/19
Tuesday

Q.1. Find the DFT & plot the magnitude & phase response of the result. $x(n) = \{1, 2, 3, 4, 4, 3, 2, 1\}$

soln:

$X(k) = \{20, -5.8 - 2.414j, 0, -0.172 - 0.414j, 0, -0.172 + 0.414j, 0, -5.8 + 2.414j\}$.

$X(0) = 20 \angle 0^\circ$

$X(1) = 6.28 \angle -157.40$

$X(2) = 0$

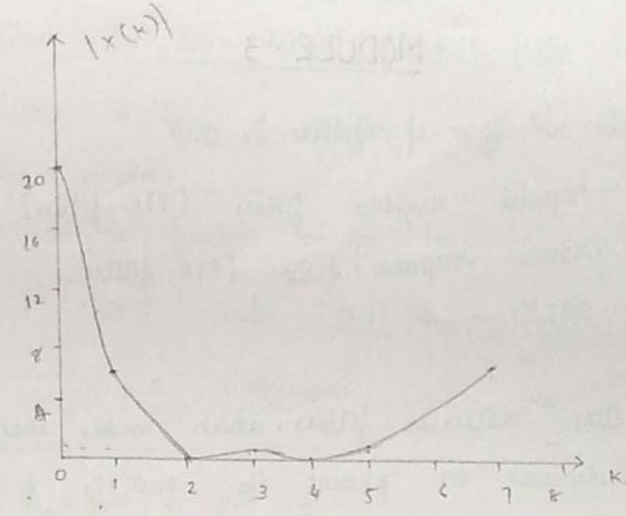
$X(3) = 0.448 \angle -112.56$

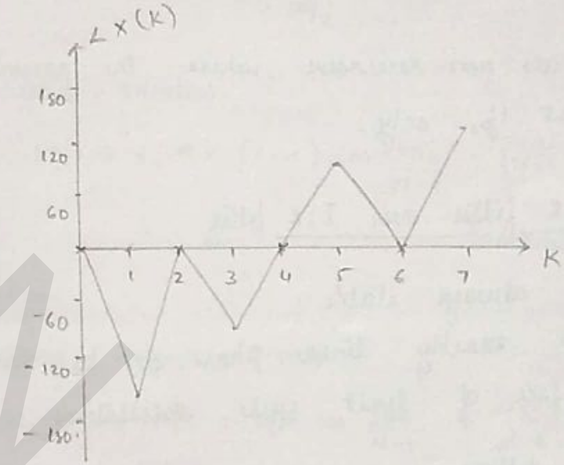
$X(4) = 0$

$X(5) = 0.448 \angle 112.56$

$X(6) = 0$

$X(7) = 6.282 \angle 157.40$





2. $x(n) = \{1, 2, 3, 4\}$

4-point DFT $= \{1, 2, 3, 4\}$

8-point DFT $= \{1, 2, 3, 4, 0, 0, 0, 0\}$

31/8/19
Saturday

MODULE - 3

There are 2 types of filters in DSP.

1. Infinite impulse response filters (IIR filter)
2. Finite impulse response filters (FIR filter).

IIR Filters

IIR filters are recursive filters, which means that the present o/p sample depends on present i/p, past i/p & past o/p.

FIR Filters

FIR filters are non-recursive, where the present o/p depends on present & past i/ps only.

Advantages of FIR filter over IIR filter

* FIR filters are always stable.
* FIR filters with exactly linear phase can be easily designed.
* FIR filters are free of limit cycle oscillations.

Disadvantages of FIR filter

* FIR filter implementation is very ~~po~~ costly.
* Memory requirements is very high.
* Execution time is very high.

Window method for the design of FIR filter

There are 3 types of windows, that we use.

1. Rectangular window.

2. The window function is gn by,

$$W_R(n) = 1 \quad ; \text{for } -\frac{(N-1)}{2} \le n \le \frac{(N-1)}{2}$$

$$0 \quad ; \text{otherwise}.$$

spectrum,

$$W_R(e^{j\omega}) = \frac{\sin \frac{\omega N}{2}}{\sin \omega/2}.$$

2. Raised cosine window.

$$W_\alpha(n) = \alpha + (1-\alpha)\cos\frac{2\pi n}{N-1} \quad ; \text{for } -\left(\frac{N-1}{2}\right) \le n \le \left(\frac{N-1}{2}\right)$$

$$= 0 \quad ; \text{otherwise}.$$

Two types:

ⓐ Hanning → put $\alpha = 0.5$

$$W_{Hn}(n) = 0.5 + 0.5\cos\frac{2\pi n}{N-1} \quad ; -\left(\frac{N-1}{2}\right) \le n \le \left(\frac{N-1}{2}\right)$$

$$= 0 \quad ; \text{otherwise}.$$

ⓑ Hamming → put $\alpha = 0.54$

$$W_{Hm}(n) = 0.54 + 0.46\cos\left(\frac{2\pi n}{N-1}\right) \quad ; -\left(\frac{N-1}{2}\right) \le n \le \left(\frac{N-1}{2}\right)$$

$$= 0 \quad ; \text{otherwise}.$$

2/9/19 Monday

Q.(1) Design an ideal low pass filter with a frequency response

$$H_d(e^{j\omega}) = 1 \quad ; -\pi/2 < \omega < \pi/2$$
$$0 \quad ; \text{otherwise.}$$

Use rectangular window with window length $N = 11$.

soln:

step 1 — Find $h_d(n)$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega.$$

$$= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} 1 \cdot e^{j\omega n}\, d\omega.$$

$$= \frac{1}{2\pi} \cdot \left(\frac{e^{j\omega n}}{jn}\right)_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{2\pi jn}\left(e^{j\frac{\pi}{2}n} - e^{-j\frac{\pi}{2}n}\right)$$

$$\because \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$\Rightarrow h_d(n) = \frac{\sin\frac{\pi}{2}n}{\pi n}$$

step 2 — Find $h_d(n)$ for all values of $-\left(\frac{N-1}{2}\right)$ to $\left(\frac{N-1}{2}\right) \Rightarrow -5$ to 5.

here $h(-n) = h(n)$ since $\sin(\theta) = \frac{\sin\theta}{\theta n}$

$$h_d(0) = \lim_{n\to 0} \frac{\sin n\pi/2}{n\pi} = \lim_{n\to 0} \frac{\sin n\pi/2}{\frac{n\pi}{2} \times 2} = \frac{1}{2}$$

$$h_d(+5) = h_d(-5) = \frac{\sin \frac{\pi}{2} \times -5}{-5 \times \pi} = 0.0636$$

$$h_d(4) = h_d(-4) = \frac{\sin \frac{-4\pi}{2}}{-4\pi} = 0$$

$$h_d(3) = h_d(-3) = \frac{\sin -3\pi/2}{-3\pi} = -0.106$$

$$h_d(2) = h_d(-2) = 0$$

$$h_d(1) = h_d(-1) = 0.3183$$

3/9/19 Tuesday

step 3 — window ~~construction~~ function. & find $h(n)$.

Window function of rectangular window, $W_R(n) = 1$.

$\Rightarrow W_R(-5) \ldots\ldots W_R(5) = 1.$

$$h(n) = h_d(n) \cdot W_R(n).$$

$h(-5) = h_d(-5) \cdot W_R(-5) = 0.0636 = h(5)$

$h(-4) = 0 = h(4)$

$h(-3) = -0.106 = h(3)$

$h(-2) = 0 = h(2)$

$h(-1) = 0.3183 = h(1)$

$h(0) = h_d(0) \cdot W_R(0) = \frac{1}{2} \cdot 1 = 1/2.$

step 4 — The transfer function of the filter.

$$H(z) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} \left[h(n)\left(z^{n} + z^{-n}\right)\right]$$

Here, $H(z) = h(0) + \sum_{n=1}^{5} \left[h(n)(z^n + z^{-n}) \right]$

$= h(0) + h(1)(z^1+z^{-1}) + h(2)(z^2+z^{-2}) + \ldots$

$\ldots + h(5)(z^5+z^{-5})$

$= \frac{1}{2} + 0.3183(z^{+1}+z^{-1}) + 0 + (-0.106(z^3+z^{-3}))$

$+ 0 + 0.0636(z^5+z^{-5})$

$= \frac{1}{2} + 0.3183(z^1+z^{-1}) - 0.106(z^3+z^{-3}) +$

$0.0636(z^5+z^{-5})$

step 5 – Transfer function of realisable filter.

$H'(z) = z^{-\left(\frac{N-1}{2}\right)} H(z)$. N = 11

Here. $H'(z) = z^{-5} \cdot H(z)$

$= z^{-5}\left[\frac{1}{2} + 0.3183(z^1+z^{-1}) - 0.106(z^3+z^{-3}) + 0.0636(z^5+z^{-5})\right]$

$= \left[\frac{1}{2}z^{-5} + 0.3183(z^{-4}+z^{-6}) - 0.106(z^{-2}+z^{-8}) + 0.0636(z^{-10}+z^0)\right]$

$= \frac{1}{2}z^{-5} + 0.3183z^{-4} + 0.3183z^{-6} - 0.106z^{-2} - 0.106z^{-8} + 0.0636z^{-10} + 0.0636$

$= 0.0636 - 0.106z^{-2} + 0.3183z^{-4} + \frac{1}{2}z^{-5} + 0.3183z^{-6} - 0.106z^{-8} + 0.0636z^{-10}$

Coefficients of the realised filter are.

$h(0) = 0.0636$	$h(5) = 0.5$
$h(1) = 0$	$h(6) = 0.3183$
$h(2) = -0.106$	$h(7) = 0$
$h(3) = 0$	$h(8) = -0.106$
$h(4) = 0.3183$	$h(9) = 0$
	$h(10) = 0.0636$

2. Use Raised cosine window.

$H_d(e^{j\omega}) = 1 ; \; -\pi/2 < \omega < \pi/2$ N = 11.

$= 0$; otherwise.

soln: $h_d(n)$ is same as that of previous question.

$h_d(n) = \frac{\sin \frac{\pi n}{2}}{\pi n}$

$h_d(0) = \frac{1}{2}$

$h_d(1) = h_d(-1) = 0.3183$

$h_d(2) = h_d(-2) = 0$

$h_d(3) = h_d(-3) = -0.106$

$h_d(4) = h_d(-4) = 0$

$h_d(5) = h_d(-5) = 0.0636$

(i)

step 3 – Here, $W_{Hn}(n) = 0.5 + 0.5\cos\frac{2\pi n}{N-1}$; $-5 \le n \le 5$

(Hanning)

$h(n) = h_d(n)\, W_{Hn}(n)$

Since cos is even fn, $\cos(-n) = \cos(n)$ $\Rightarrow W_{Hn}(n) = W_{Hn}(-n)$

$h(0) = \frac{1}{2} \cdot (0.5 + 0.5\cos 0) = \frac{1}{2}$

$h(1) = 0.3183 \times \left(0.5 + 0.5\cos\frac{2\pi}{10}\right) = 0.2879 = h(-1)$

$h(2) = 0 = h(-2)$

$h(3) = -0.106 \times \left(0.5 + 0.5 \cos \frac{2\pi}{10} \times 3\right) = -0.0366 = h(-3)$

$h(4) = 0 = h(-4)$

$h(5) = 0.0636 \left(0.5 + 0.5 \cos \frac{2\pi}{10} \times 5\right) = 0.$

step 4 – Transfer function of filter.

$$H(z) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} \left[h(n)\,(z^n + z^{-n})\right]$$

$$= \frac{1}{2} + 0.2879\,(z^1 + z^{-1}) + 0 + -0.0366\,(z^3 + z^{-3}) + 0 + 0$$

step 5 – Transfer function of realisable filter

$$H'(z) = z^{-\left(\frac{N-1}{2}\right)} H(z)$$

$$= z^{-5} H(z)$$

$$= \frac{1}{2} z^{-5} + 0.2879\, z^{-4} + 0.2819\, z^{-6} - 0.0366 z^{-2} - 0.0366\, z^{-8}$$

$$= -0.0366\, z^{-2} + 0.2879 z^{-4} + \frac{1}{2} z^{-5} + 0.2879 z^{-6} - 0.0366 z^{-8}$$

Coefficients of $H'(z)$

$h(0) = 0$	$h(5) = \frac{1}{2}$
$h(1) = 0$	$h(6) = 0.2879$
$h(2) = -0.0366$	$h(7) = 0$
$h(3) = 0$	$h(8) = -0.0366$
$h(4) = 0.2879$	$h(9) = 0$
	$h(10) = 0$

(ii) Hamming.

step 3: $W_{Hm}(n) = 0.54 + 0.46 \cos\left(\frac{2\pi n}{N-1}\right)$; $-5 \le n \le 5$

$h(n) = h_d(n)\, W_{Hm}(n)$

$h(0) = h_d(0)\, W_{Hm}(0) = \frac{1}{2}\,(0.54 + 0.46 \cos 0) = \frac{1}{2}$

$h(1) = 0.3183 \cdot W_{Hm}(1) = 0.2903 = h(-1)$

$h(2) = 0 = h(-2)$

$h(3) = -0.106\, \left(0.54 + 0.46 \cos \frac{2\pi}{10} \times 3\right) = -0.0421 = h(-3)$

$h(4) = 0 = h(-4)$

$h(5) = 0.0636 \times W_{Hm}(5) = 5.088 \times 10^{-3} = h(-5)$

step 4 – $H(z) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} \left[h(n)\,(z^n + z^{-n})\right]$

$$= \frac{1}{2} + h(1)\,[z^1 + z^{-1}] + h(2)\,[z^2 + z^{-2}] + \ldots + h(5)\,[z^5 + z^{-5}]$$

$$= \frac{1}{2} + 0.3183\,[z^1 + z^{-1}] + 0 + -0.0421\,(z^3 + z^{-3}) + 0 + 5.088 \times 10^{-3}(z^5 + z^{-5})$$

step 5 – $H'(z) = z^{-\left(\frac{N-1}{2}\right)} H(z)$

$$= z^{-5} H(z)$$

$$= \frac{1}{2} z^{-5} + 0.3183\,[z^{-4} + z^{-6}] - 0.0421\,[z^{-2} + z^{-8}] + 5.088 \times 10^{-3}\,[z^0 + z^{-10}]$$

$$= 5.088 \times 10^{-3} - 0.0421 z^{-2} + 0.3183 z^{-4} + 0.5 z^{-5} + 0.3183 z^{-6} - 0.0421 z^{-8} + 5.088 \times 10^{-3} z^{-10}$$

Coefficients of $H'(z)$

$h(0) = 5.088\times10^{-3}$ $\quad$ $h(6) = 0.3183$

$h(1) = 0$ $\quad$ $h(7) = 0$

$h(2) = -0.0421$ $\quad$ $h(8) = -0.0421$

$h(3) = 0$ $\quad$ $h(9) = 0$

$h(4) = 0.3183$ $\quad$ $h(10) = 5.088\times10^{-3}$

$h(5) = 0.5$

4/9/19 Wednesday

* Q1. Design a linear phase FIR low pass filter having length $N = 15$ & cut off freq $\omega_c = \pi/6$. Use hamming window.

soln: As the filter is a low pass filter,

$$H_d(e^{j\omega}) = 1 \; ; \; |\omega| \le \pi/6$$
$$0 \; ; \; \text{otherwise}.$$

$$|\omega| \le \pi/6 \Rightarrow -\pi/6 < \omega < \pi/6$$

$h_d(n) = ?$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}).e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi/6}^{\pi/6} 1.\,e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\left(\frac{e^{j\omega n}}{jn}\right)_{-\pi/6}^{\pi/6} = \frac{1}{2\pi nj}\left(e^{\frac{\pi}{6}jn} - e^{-\frac{\pi}{6}jn}\right)$$

$$= \frac{\sin n\pi/6}{n\pi}$$

Remaining steps are same as previous question

For HPF,

$$H_d(e^{j\omega}) = 1 \; ; \; \pi/6 \le \omega < \pi$$
$$0 \; ; \; \text{otherwise}$$

* 2. Prove that if z_1 is a zero of linear phase FIR filter then $\frac{1}{z_1}$ is also a zero.

soln: We know,

$$H(z) = \sum_{k=0}^{m} h_k z^{-k}$$ (z-transform with impulse response length $m+1$)

$$H(1/z) = H(z^{-1}) = \sum_{k=0}^{m} h_k z^{k}$$

$$= \sum_{k=0}^{m} h_k z^{k} z^{-m} z^{m}$$

$$= z^{m}\sum_{k=0}^{m} h_k z^{-(m-k)}$$

$$= z^{m}\sum_{k=0}^{m} h_{m-n} z^{-n}$$

where, $n = m-k$

If h_k is symmetric, $h_k = h_{m-k}$

$$\Rightarrow H(z^{-1}) = z^{m} H(z)$$
$$H(1/z) = z^{m} H(z)$$

If z_1 is a zero of $H(z) \Rightarrow H(z_1) = 0$

$$H(1/z_1) = z_1^{m} H(z_1)$$

If $H(z_1) = 0$, then above equ. gives,

$$H(1/z_1) = 0$$

3. Find the response of the signal $x(n) = 2\cos(\pi/2).n$ when applied to a FIR filter whose impulse response is $h(n) = \{1,3,1\}$.

soln:

$x(n) = 2\cos\frac{n\pi}{2}$. $\quad h(n) = \{1,3,1\}$

Since, $h(n)$ consists of 3 elements, put $n = 0,1,2$.

$x(0) = 2$

$x(1) = 0 \qquad x(n) = \{2,0,-2\}$

$x(2) = -2$.

$y(n) = x(n) * h(n)$

	1	3	1
2	2	6	2
0	0	0	0
-2	-2	-6	-2

$y(n) = \{2, 6, 0, -6, -2\}$

4. A second order linear phase FIR filter has a zero at $z = 1/2$. Find the magnitude of the filter.

soln. If z_1 is a root of the linear phase FIR filter then $1/z_1$ is also a root. So the roots of the gn filter are,

$z = 1/2$ & $z = 2$.

$\Rightarrow (z - 1/2)(z-2) = 0$

$z^2 - 2z - \frac{z}{2} + 1 = 0 \quad \Rightarrow z^2 - \frac{5z}{2} + 1 = 0$

$H(z) = z^2 - \frac{5}{2}z + 1$

magnitude, $h(n) = \{1, -5/2, 1\}$.

11/9/19
Tuesday

Linear phase FIR filters

An LTI system performs a type of filtering among the various freq components at its input. The nature of this filtering is determined by the freq response $H(\omega)$, which in ~~turn~~ turn depends upon the system parameter like the filter coefficients. Thus by proper selection of coeff we can design freq selective filters that pass signals with freq components in some bands while attenuating freq components in other bands. We can write the expression for o/p signal spectrum as

$$Y(\omega) = H(\omega) \cdot X(\omega)$$

Suppose we are considering a signal $x(n)$ with freq content in a band of freq ~~rope~~ $\omega_1 < \omega < \omega_2$. Hence we can write,

$$X(\omega) = 0 \quad ; \text{for } \omega \le \omega_1 \ \& \ \omega \ge \omega_2.$$

Let this signal is passed through a filter with freq response

$$H(\omega) = \begin{cases} C e^{-j\omega\alpha} & ; \omega_1 < \omega < \omega_2 \\ 0 & ; \text{otherwise.} \end{cases}$$

$C, \alpha \rightarrow$ +ve constants.

$$Y(\omega) = X(\omega) \cdot H(\omega)$$
$$= X(\omega) \cdot Ce^{-j\omega\alpha}$$
$$= CX(\omega) \cdot e^{-j\omega\alpha} \quad ; \omega_1 < \omega < \omega_2$$

By time shifting property,

Fourier Transform $FT[x(n-\alpha)] = X(\omega) \cdot e^{-j\omega\alpha}$

$$\Rightarrow Y(\omega) = C \cdot FT[x(n-\alpha)]$$

By taking inverse,

$$y(n) = Cx(n-\alpha)$$

$H(\omega)$ is a complex quantity.

Here, $H(\omega) = |H(\omega)| \cdot \angle H(\omega) = C \cdot e^{-j\omega\alpha}$ — ①

Let $\angle H(\omega) = \theta(\omega)$

So comparing equ. ① we get,

$$|H(\omega)| = C$$

$$\angle H(\omega) = \theta(\omega) = -\alpha\omega$$

Here we can say that amplitude is constant and phase is a linear function of frequency. So we call the filter as linear phase filter.

The signal delay as a function of freq can be obtained as $\tau(\omega) = -\frac{d\theta(\omega)}{d\omega} = \alpha$

23/9/19 Monday – **Symmetric and Anti-symmetric filter**

Let $h(n)$ be a causal finite duration sequence defined over the interval $0 \le n \le N-1$ and let the samples of $h(n)$ be real. The Fourier transform of $h(n)$ is

$$H(\omega) = \sum_{n=0}^{N-1} h(n) \cdot e^{-j\omega n} \quad — ①$$

$H(\omega)$ can be expressed as magnitude & phase function as

$$H(\omega) = \pm |H(\omega)| \cdot e^{j\theta(\omega)} \quad — ②$$

If $h(n)$ is real, then magnitude function is symmetric.

$$\Rightarrow |H(\omega)| = |H(-\omega)|$$

Phase function is anti-symmetric.

$$\Rightarrow |\theta(\omega)| = -|\theta(-\omega)|$$

To get exactly linear phase

$\theta(\omega) \propto \omega$ or

$$\theta(\omega) = -\alpha\omega \quad — ③$$

Put ③ in ②

$$H(\omega) = \pm |H(\omega)| e^{-j\alpha\omega} \quad — ④$$

from ① & ④,

$$\sum_{n=0}^{N-1} h(n) \cdot e^{-j\omega n} = \pm |H(\omega)| e^{-j\alpha\omega}$$

$$\sum_{n=0}^{N-1} h(n)(\cos\omega n - j\sin\omega n) = \pm |H(\omega)| [\cos\alpha\omega - j\sin\alpha\omega]$$

Equating real & imaginary parts,

$$\pm |H(\omega)| \cos\alpha\omega = \sum_{n=0}^{N-1} h(n)\cos\omega n \quad — ⑤$$

$$\pm|H(\omega)| \sin \alpha\omega = \sum_{n=0}^{N-1} h(n) \sin \omega n \quad — ⑥.$$

⑥/⑤,

$$\frac{\sin \alpha\omega}{\cos \alpha\omega} = \frac{\sum_{n=0}^{N-1} h(n) \sin \omega n}{\sum_{n=0}^{N-1} h(n) \cos \omega n}$$

Cross multiply,

$$\sin \alpha\omega \sum_{n=0}^{N-1} h(n) \cos \omega n = \cos \alpha\omega \sum_{n=0}^{N-1} h(n) \sin \omega n$$

$$\sum_{n=0}^{N-1} h(n) \left[\sin \alpha\omega \cos \omega n - \cos \alpha\omega \sin \omega n \right] = 0$$

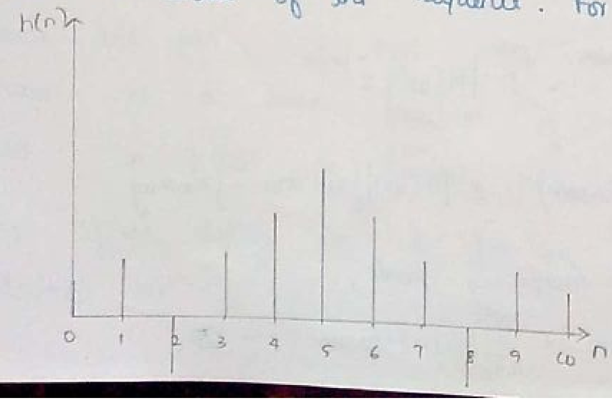
$$\sum_{n=0}^{N-1} h(n) \left[\sin (\alpha - n)\omega \right] = 0$$

Solution to above equation exists for

$$\alpha = \frac{N-1}{2} \quad \& \quad h(n) = h(N-1-n)$$

Symmetric filter

For the condition given above, we can observe that for ~~for~~ different of α and N, the impulse response is symmetric about the centre of the sequence. For example, if $N = 11$ & $\alpha = 5$



~~$H(\omega) = \pm |H(\omega)| e^{j(\beta - \alpha\omega)}$~~

~~Possible solution for it~~

Such filters will have constant group delay & constant phase delay.

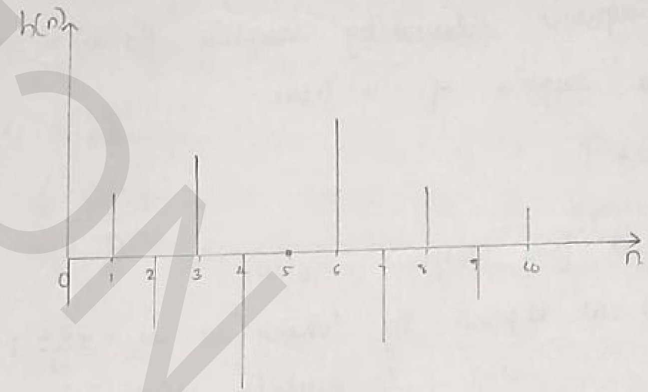
Anti-symmetric filter

If only constant group delay is required then,

$$H(\omega) = \pm |H(\omega)| e^{j(\beta - \alpha\omega)}$$

Possible solution for this is

$$\alpha = \frac{N-1}{2}, \quad \beta = \pm \pi/2, \quad h(n) = -h(N-1-n)$$



Frequency response of linear phase FIR filters

Depending on the value of N (odd or even) & type of symmetry of filter impulse response, there are four possible types of linear phase FIR filter.

1. Symmetric impulse response when N is odd.
2. symmetric impulse response when N is even,

3. Anti-symmetric impulse response when N is odd
4. Anti-symmetric impulse response when N is even.

24/9/19
Tuesday.

Frequency Sampling method

In this method, ideal freq response is sampled at sufficient no. of points (N points). These samples are DFT coeff. of the impulse response of the filter. Hence the impulse response of the filter is determined by taking the inverse DFT.

Here, $H_d(\omega)$ = Ideal desired freq response.

$H(k)$ = DFT sequence obtained by sampling $H_d(\omega)$

$h(n)$ = impulse response of FIR filter.

Type I design

step 1 : Choose the ideal freq response $H_d(\omega)$

step 2 : Sample $H_d(\omega)$ at N points by taking $\omega = \omega_k = \frac{2\pi k}{N}$; where $k = 0, 1, 2, 3, \ldots N-1$. to generate $H(k)$.

$$\therefore \; H(k) = H_d(\omega)\Big|_{\omega = \frac{2\pi k}{N}}$$

step 3 : Compute h(n) using the following equation.

$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{\frac{N-1}{2}} Re\left(H(k)\, e^{\frac{j2\pi nk}{N}}\right)\right] \rightarrow N \text{ is odd.}$$

$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{\frac{N}{2}-1} Re\left(H(k)\, e^{\frac{j2\pi nk}{N}}\right)\right] \rightarrow N \text{ is even.}$$

step 4 : The ZT of the impulse response

$$H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n}.$$

Type II design

step 1 : Same as above

step 2 : $H(k) = H_d(\omega)\Big|_{\omega = \frac{2\pi(2k+1)}{N}}$

step 3 : $h(n) = \frac{2}{N}\sum_{k=0}^{\frac{N-3}{2}} Re\left[H(k)\, e^{\frac{jn\pi(2k+1)}{N}}\right] \rightarrow N$ is odd.

$h(n) = \frac{2}{N} \cdot 2\sum_{k=0}^{\frac{N}{2}-1} Re\left[H(k) \cdot e^{\frac{jn\pi(2k+1)}{N}}\right] \rightarrow N$ is even.

step 4 : same as above.

Q.1. Determine the coeff. of a linear phase FIR filter of length N = 15 which has a symmetric unit sample response and a freq response that satisfies the condition;

$$H\left(\frac{2\pi k}{15}\right) = \begin{cases} 1 & ; k = 0, 1, 2, 3 \\ 0.4 & ; k = 4 \\ 0 & ; k = 5, 6, 7. \end{cases}$$

$h(n) = h(N-1-n)$

soln: $\alpha = \frac{N-1}{2} = \frac{15-1}{2} = 7$

Multiply the terms of $H_d(\omega)$ with $e^{-\alpha j\omega}$

$$H_d(\omega) = \begin{cases} e^{-j\alpha\omega} & ; k = 0, 1, 2, 3 \\ 0.4\, e^{-j\alpha\omega} & ; k = 4 \\ 0 & ; k = 5, 6, 7 \end{cases}$$

$H(K) = ?$

$k=0 \Rightarrow H_d(\omega) = e^{-j\alpha\omega}$

$H(0) = e^{-j\alpha \frac{2\pi k}{15}}\Big|_{k=0} = e^0 = 1$

$k=1 \rightarrow e^{-j\alpha\omega}$

$H(1) = e^{-j \times 7 \times \frac{2\pi}{15} \times k}\Big|_{k=1} = e^{-j\frac{14\pi}{15}} = e^{-j2.93}$

$k=2 \rightarrow e^{-j\alpha\omega}$

$H(2) = e^{-j\frac{7 \times 2\pi k}{15}}\Big|_{k=2} = e^{-j\frac{28}{15}\pi} = e^{-5.86j}$

$k=3 \rightarrow e^{-j\alpha\omega}$

$H(3) = e^{-j\frac{7 \times 2\pi \times 3}{15}} = e^{-8.79j}$

$k=4 \rightarrow 0.4e^{-j\alpha\omega}$

$H(4) = 0.4e^{-j \times \frac{7 \times 2\pi \times 4}{15}} = 0.4e^{-11.72j}$

$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{\frac{N-1}{2}} Re\left[H(K)\, e^{\frac{j2\pi nk}{N}}\right]\right]$; N is odd.

$= \frac{1}{15}\left[1 + 2\sum_{k=1}^{7} Re\left[H(k)\, e^{\frac{j2\pi nk}{15}}\right]\right]$

$= \frac{1}{15}\left[1 + 2\left[\sum_{k=1}^{3} Re\left[H(K)\, e^{\frac{j2\pi nk}{15}}\right] + \sum_{k=4}^{7} Re\left[H(K)\, e^{\frac{j2\pi nk}{15}}\right]\right]\right]$

$= \frac{1}{15}\left[1 + 2\left[\sum_{k=1}^{3} Re\left(e^{-j \times 7 \times \frac{2\pi k}{15}} \cdot e^{\frac{j2\pi nk}{15}}\right) + \sum_{k=4}^{7} Re\left(0.4e^{-j7 \times \frac{2\pi k}{15}} \cdot e^{\frac{j2\pi nk}{15}}\right)\right]\right]$

$= \frac{1}{15}\left[1 + 2\left(\sum_{k=1}^{3} Re\left(e^{j\frac{2\pi k}{15}(n-7)}\right) + \sum_{k=4}^{7} Re\left(0.4\, e^{\frac{j2\pi k}{15}(n-7)}\right)\right)\right]$

$= \frac{1}{15}\left[1 + 2\left(\sum_{k=1}^{3} \cos\frac{2\pi k}{15}(n-7) + 0.4\cos\frac{8\pi}{15}(n-7)\right)\right]$

$h(n) = \frac{1}{15}\left[1 + 2\left(\cos\frac{2\pi}{15}(n-7) + \cos\frac{4\pi}{15}(n-7) + \cos\frac{6\pi}{15}(n-7)\right) + 0.8\left(\cos\frac{8\pi}{15}(n-7)\right)\right]$

$h(0) = -0.2119$

$h(1) = -0.0291$

$h(2) = 0.6$

$h(3) = 0.1835$

$h(4) = -1.3708$

$h(5) = -0.2713$

$h(6) = 4.699$

$h(7) = 7.8$

$h(8) = h(15-1-8) = h(6)$

$h(9) = h(5)$

$h(10) = h(4)$

$h(11) = h(3)$

$h(12) = h(2)$

$h(13) = h(1)$

$h(14) = h(0)$

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

$$= \sum_{n=0}^{14} h(n).z^{-n}$$

$$= -0.2119(1+z^{-14}) - 0.0291(z^{-1}+z^{-13}) + 0.6(z^{-2}+z^{-12}) + 0.1835(z^{-3}+z^{-11}) - 1.3708(z^{-4}+z^{-10}) - 0.2713(z^{-5}+z^{-9}) + 4.699(z^{-6}+z^{-8}) + 7.8(z^{-7})$$

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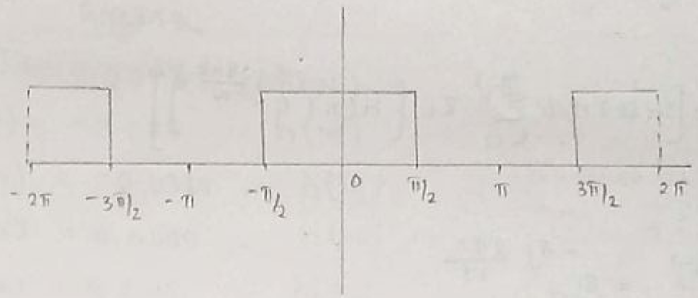
Q. Design a LPF with a cut-off frequency of 5π/π/2 sec. Take N = 17 and use freq sampling method.



$$H_d(\omega) = \begin{cases} e^{-\alpha j\omega} & ; 0 < \omega < \pi/2 \quad (0 < \omega < 0.5\pi) \\ 0 & ; \pi/2 < \omega < 3\pi/2 \quad (0.5\pi < \omega < 1.5\pi) \\ e^{-\alpha j\omega} & ; 3\pi/2 < \omega < 2\pi \quad (1.5\pi < \omega < 2\pi) \end{cases}$$

$$\omega_k = \frac{2\pi K}{N} = \frac{2\pi K}{17} \quad ; K = 0, 1, \ldots 16.$$

$\omega_0 = 0$	$\omega_8 = 0.94\pi$
$\omega_1 = 0.117\pi$	$\omega_9 = 1.058\pi$
$\omega_2 = 0.23\pi$	$\omega_{10} = 1.17\pi$
$\omega_3 = 0.35\pi$	$\omega_{11} = 1.29\pi$
$\omega_4 = 0.47\pi$	$\omega_{12} = 1.41\pi$
$\omega_5 = 0.58\pi$	$\omega_{13} = 1.52\pi$
$\omega_6 = 0.7\pi$	$\omega_{14} = 1.64\pi$
$\omega_7 = 0.82\pi$	$\omega_{15} = 1.76\pi$
	$\omega_{16} = 1.88\pi$

$$H_d(\omega) = \begin{cases} e^{-j\alpha\omega} & ; K = 0-4 \\ 0 & ; K = 5-12 \\ e^{-j\alpha\omega} & ; K = 13-16 \end{cases}$$

$\alpha = \frac{N-1}{2} = \frac{17-1}{2} = 8$

$H(K) \approx 8$

$K=0 \Rightarrow 8^{-\alpha j\omega}$

$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{K=1}^{\frac{N-1}{2}} Re\left[H(K)\, e^{\frac{j2\pi nk}{N}}\right]\right]$$

$H(0) = ?$

$K=0 \rightarrow e^{-\alpha j\omega} = e^{-8j\frac{2\pi K}{17}}$

$= e^{-0} = 1.$

$$\Rightarrow h(n) = \frac{1}{17}\left[1 + 2\sum_{K=1}^{8} Re\left(H(K)\, e^{\frac{j2\pi nk}{17}}\right)\right]$$

$= \frac{1}{17}\left[1 + 2\sum_{K=1}^{8} Re\left(\right.\right.$

$$= \frac{1}{17}\left[1 + 2\left(\sum_{K=1}^{4} Re\left(H(K)\, e^{\frac{j2\pi nk}{17}}\right) + \sum_{K=5}^{8} Re\left(H(K)\, e^{\frac{j2\pi nk}{17}}\right)\right)\right]$$

$= 0 \;;\; K = 5-12$

$$= \frac{1}{17}\left[1 + 2\sum_{K=1}^{4} Re\left(H(K)\, e^{\frac{j2\pi nk}{17}}\right)\right]$$

$$= \frac{1}{17}\left[1 + 2\sum_{K=1}^{4} Re\left(e^{-j\cdot 8\cdot\frac{2\pi K}{17}} \cdot e^{\frac{j2\pi nk}{17}}\right)\right]$$

$= \frac{1}{17}\left[1 + 2\sum_{K=1}^{4}\left(\cos\frac{16\pi K}{17}\right.\right.$

$$= \frac{1}{17}\left[1 + 2\sum_{K=1}^{4} Re\left(e^{\frac{j2\pi K}{17}(n-8)}\right)\right]$$

$$= \frac{1}{17}\left[1 + 2\sum_{K=1}^{4} \cos\frac{2\pi K}{17}(n-8)\right]$$

$$= \frac{1}{17}\left[1 + 2\left(\cos\frac{2\pi}{17}(n-8) + \cos\frac{4\pi}{17}(n-8) + \cos\frac{6\pi}{17}(n-8) + \cos\frac{8\pi}{17}(n-8)\right)\right]$$

$h(0) = 0.0398 = h(16)$

$h(1) = -0.048 = h(15)$

$h(2) = -0.0345 = h(14)$

$h(3) = 0.0659 = h(13)$

$h(4) = 0.0315 = h(12)$

$h(5) = -0.1075 = h(11)$

$h(6) = -0.0299 = h(10)$

$h(7) = 0.3187 = h(9)$

$h(8) = 0.5294$

$h(n) = h(N-1-n)$

$n = 9 \Rightarrow h(9) = h(17-1-9)$

$h(9) = h(7)$

$$H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n}$$

$$= \sum_{n=0}^{16} h(n)\, z^{-n}$$

$= (0.0398 - 0.048\, z^{-1} - 0.0345$

$$= \left(0.0398\,(1 + z^{-16}) - 0.048\,(z^{-1} + z^{-15}) - 0.0345\,(z^{-2} + z^{-14}) + 0.0659\,(z^{-3} + z^{-13}) + 0.0315\,(z^{-4} + z^{-12}) - 0.1075\,(z^{-5} + z^{-11}) - 0.0299\,(z^{-6} + z^{-10}) + 0.3187\,(z^{-7} + z^{-9}) + 0.5294\, z^{-8}\right)$$

Q. A LPF has a desired frequency response *

$$H_d(\omega) = \begin{cases} e^{-j3\omega} & ; 0 < \omega < \pi/2 \\ 0 & ; \pi/2 < \omega < \pi \end{cases}$$

Determine $h(n)$ based on frequency sampling technique.

Soln:

$K = \frac{N-1}{2} = 3$.

$N-1 = 6 \Rightarrow N = 7$.

$\omega_k = \frac{2\pi K}{N} = \frac{2\pi K}{7}$; $K = 0, 1, \ldots 6$

$\omega_0 = 0$ $\qquad \omega_4 = 1.143\pi$

$\omega_1 = 0.285\pi$ $\qquad \omega_5 = 1.428\pi$

$\omega_2 = 0.571\pi$ $\qquad \omega_6 = 1.714\pi$

$\omega_3 = 0.857\pi$

$$H_d(\omega) = \begin{cases} e^{-j3\omega} & ; 0 < \omega < 0.5\pi \Rightarrow K = 0-1 \\ 0 & ; 0.5\pi < \omega < \pi \Rightarrow K = 2-3 \end{cases}$$

$$\Rightarrow h(n) = \frac{1}{N}\left[H(0) + 2\sum_{K=1}^{\frac{N-1}{2}} Re\left[H(K)\, e^{\frac{j2\pi nk}{N}}\right]\right]$$

$H(0) = ?$

$K = 0 \rightarrow e^{-j3\omega} \Rightarrow H(0) = e^{-j3 \cdot \frac{2\pi K}{7}}$

$= e^{-0}$

$= 1$

$$\Rightarrow h(n) = \frac{1}{7}\left[1 + 2\sum_{K=1}^{3} Re\left(H(K) \cdot e^{j\frac{2\pi nk}{7}}\right)\right]$$

$$= \frac{1}{7}\left[1 + 2\left(Re\left(H(K) \cdot e^{j\frac{2\pi nk}{7}}\right)\Big|_{K=1} + \sum_{K=2}^{3} Re\left(\underbrace{H(K)}_{=0\ ;\ K\to 2-3} \cdot e^{j\frac{2\pi nk}{7}}\right)\right)\right]$$

$$= \frac{1}{7}\left[1 + 2\, Re\left(e^{-j3\frac{2\pi K}{7}} \cdot e^{j\frac{2\pi nk}{7}}\right)_{K=1}\right]$$

$$= \frac{1}{7}\left[1 + 2 \cdot Re\left(e^{j\frac{2\pi K}{7}(n-3)}\right)_{K=1}\right]$$

$$= \frac{1}{7}\left[1 + 2 \cdot Re\left(e^{j\frac{2\pi}{7}(n-3)}\right)\right]$$

$$= \frac{1}{7}\left[1 + 2 \cdot \cos\frac{2\pi}{7}(n-3)\right]$$

$h(0) = -0.1145 = h(6)$ $\qquad h(n) = h(N-1-n)$

$h(1) = 0.0792 = h(5)$ $\qquad n = 0-6$.

$h(2) = 0.321 = h(4)$ $\qquad h(6) = h(7-1-6)$

$h(3) = 0.4285$ $\qquad h(6) = h(0)$

$$H(z) = \sum_{n=0}^{N-1} h(n) \cdot z^{-n} = \sum_{n=0}^{6} h(n) \cdot z^{-n}$$

$$= \left[-0.1145(1 + z^{-6}) + 0.0792(z^{-1} + z^{-5}) + 0.321(z^{-2} + z^{-4}) + 0.4285(z^{-3})\right]$$

01/10/19 Tuesday

MODULE - 4

Design of IIR Filters

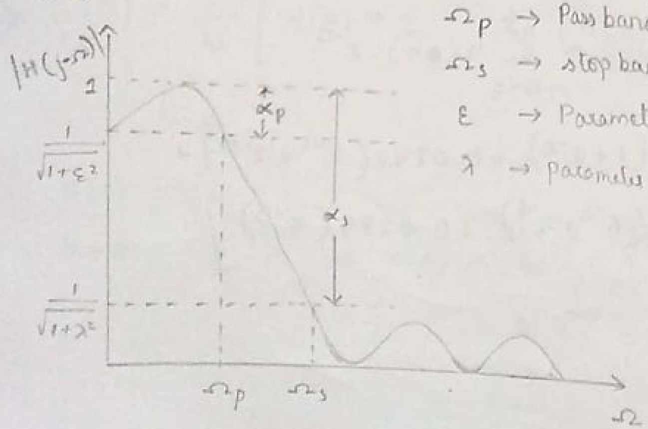
Design of digital filter from analog filter.

The most common technique used for designing IIR digital filter is known as indirect method where we first design an analog prototype filter & then transforming it into digital filter. Following steps are used for the design :

1. Map the ~~desig~~ desired digital filter specification into those for an analog filter.
2. Derive the analog filter transfer function.
3. Transform the analog filter transfer function into equivalent digital filter transfer function.

Butterworth filter design.

LPF



$\Omega_p \to$ Passband frequency

$\Omega_s \to$ stop band frequency

$\varepsilon \to$ Parameter specifying allowable passband

$\lambda \to$ parameter specifying allowable stop band

Steps to design an analog butterworth LPF:

1) From the gn specification obtain the value of N. N is gn by,

$$N \geq \frac{\log \sqrt{\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}}{\log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$$N \geq \frac{\log\left(\frac{\lambda}{\varepsilon}\right)}{\log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

where, $\varepsilon = (10^{0.1\alpha_p} - 1)^{0.5}$ & $\lambda = (10^{0.1\alpha_s} - 1)^{0.5}$.

α_p & α_s are the passband and stop band attenuation in dB respectively.

2) Round off N to the next highest integer

3). Find the transfer function H(s) for $\Omega_c = 1$ rad/s for the obtained value of N.

4) Calculate the value of cut-off freq Ω_c

$$\Omega_c = \frac{\Omega_p}{(10^{0.1\alpha_p} - 1)^{1/2N}} = \frac{\Omega_s}{(10^{0.1\alpha_s} - 1)^{\frac{1}{2N}}} \quad \text{OR} \quad \Omega_c = \frac{\Omega_p}{\varepsilon^{1/N}} = \frac{\Omega_s}{\lambda^{1/N}}$$

5) Find the transfer function $H_a(s)$ by subst. $\frac{s}{\Omega_c}$ for s.

Q.1. Design an analog butterworth filter that has a -2dB passband attenuation at a freq of 20 rad/s & atleast -10dB stop band attenuation at 30 rad/s.

soln :

$\alpha_p = 2dB$. $\alpha_s = 10dB$

$\Omega_p = 20$ rad/s $\Omega_s = 30$ rad/s.

$$N \geq \frac{\log\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log\left(\frac{\Omega_s}{\Omega_p}\right)} = \quad N \geq 3.37$$

$\underline{N = 4}$

N	Denominator of H(s)
1	$s+1$
2.	$s^2+\sqrt{2}s+1$
3	$(s+1)(s^2+s+1)$
4	$(s^2+0.76537s+1)(s^2+1.84775s+1)$
5	$(s+1)(s^2+0.618033s+1)(s^2+1.618033s+1)$

N-4

$$H(s) = \frac{1}{(s^2+0.76537s+1)(s^2+1.8477s+1)}$$

$$\Omega_c = \frac{\Omega_p}{(10^{0.1\alpha_p}-1)^{1/2N}} = 21.3867 \approx 21.387$$

Put $s = 21.387$

$$H(s) = \frac{1}{(21.387^2+0.76537\times21.387+1)(21.387^2+1.8477\times21.387+1)}$$

=

Put $s = \frac{s}{\Omega_c}$; $H_a(s) = \frac{1}{\left(\left(\frac{s}{21.387}\right)^2+\frac{0.76535 s}{21.387}+1\right)\left(\left(\frac{s}{21.387}\right)^2+\frac{1.847s}{21.387}+1\right)}$

$$H_a(s) = \frac{21.387^2 \times 21.387^2}{\left(s^2+0.76535s\times21.387+\frac{21.387^2}{21.387^2}\right)\left(s^2+1.847s\times21.387+\frac{21.387^2}{21.387^2}\right)}$$

$$= \frac{21.387^4}{(s^2+16.37s+21.387^2)(s^2+39.50s+21.387^2)}$$

$$H_a(s) = \frac{209218.2079}{(s^2+16.37s+457.4)(s^2+39.5s+457.4)}$$

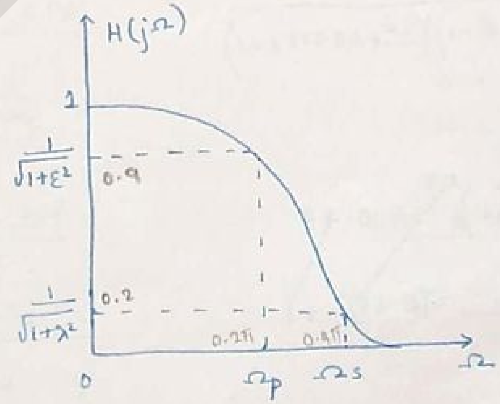
5/10/19
Saturday.

2. For the gn specifications design an analog butterworth filter.

$0.9 \leq |H(j\Omega)| \leq 1$ for $0 \leq \Omega \leq 0.2\pi$

$|H(j\Omega)| \leq 0.2$ for $0.4\pi \leq \Omega \leq \pi$

soln :



$\Omega_p = 0.2\pi$

$\Omega_s = 0.4\pi$

$$\Rightarrow \frac{1}{\sqrt{1+\varepsilon^2}} = 0.9 \Rightarrow \frac{1}{0.81} = 1+\varepsilon^2$$

$$\varepsilon^2 = \frac{1}{0.81} - 1$$

$$\varepsilon^2 = 0.23 \Rightarrow \varepsilon = 0.48$$

$$\Rightarrow \frac{1}{\sqrt{1+\lambda^2}} = 0.2$$

$$1+\lambda^2 = \frac{1}{0.04} \Rightarrow \lambda^2 = \frac{1}{0.04} - 1$$

$$\lambda = \sqrt{24}$$

$$\lambda = 4.899$$

$$N \geq \frac{\log\left(\frac{\lambda}{\varepsilon}\right)}{\log\left(\frac{\Omega_s}{\Omega_p}\right)} \Rightarrow N \geq \frac{\log\left(\frac{4.899}{0.48}\right)}{\log\left(\frac{0.4\pi}{0.2\pi}\right)}$$

$$N \geq 3.35 \Rightarrow N = 4$$

$$\Rightarrow H(s) = \frac{1}{(s^2+0.76537s+1)(s^2+1.8477s+1)}$$

to find ~~α_p~~ α_p,

$$\varepsilon = (10^{0.1\alpha_p} - 1)^{0.5} = 0.48$$

$$10^{0.1\alpha_p} = (0.48)^2 + 1$$

to find $\Omega_c = \frac{\Omega_p}{\varepsilon^{1/N}} = \frac{0.2\pi}{(0.48)^{1/4}} = 0.75$

put $s = \frac{s}{\Omega_c}$

$$\Rightarrow H_a(s) = \frac{1}{\left(\left(\frac{s}{0.75}\right)^2 + 0.76537\frac{s}{0.75} + 1\right)\left(\left(\frac{s}{0.75}\right)^2 + 1.8477\frac{s}{0.75} + 1\right)}$$

$$= \frac{0.75^2 \times 0.75^2}{(s^2 + 0.574s + 0.75^2)(s^2 + 1.385s + 0.75^2)}$$

$$H_a(s) = \frac{0.316}{(s^2+0.574s+0.5625)(s^2+1.385s+0.5625)}$$

Frequency Transformation in analog domain

LPF to LPF

$$s \rightarrow \frac{s}{\Omega_c}$$

LPF to HPF

$$s \rightarrow \frac{\Omega_c}{s}$$

LPF to BPF

$$s \rightarrow \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)}$$

LPF to BSF

$$s \rightarrow \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_l \Omega_u}$$

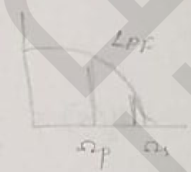
* Q. 1. For the gn specifications $\alpha_p = 3dB$, $\alpha_s = 15 dB$, $\Omega_p = 1000$ rad/s & $\Omega_s = 500$ rad/s design a HPF.

soln: HPF ⇒ $\alpha_p = 3dB$; $\alpha_s = 15dB$; $\Omega_p = 1000$ rad/s ; $\Omega_s = 500$ rad/s

According to the gn specification for a HPF, $\Omega_p = 1000$ rad/s & $\Omega_s = 500$ rad/s, so converting that to a LPF,

LPF $\begin{cases} \Omega_p = 500 \text{ rad/s} \\ \Omega_s = 1000 \text{ rad/s} \end{cases}$

And, α_p & α_s → no change.



* Since, here $\boxed{\alpha_p = 3 \text{ dB}}$ (cut-off gain),

$\boxed{\Omega_p = \Omega_c}$ in both HPF & LPF

$$N \geq \frac{\log \sqrt{\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}}{\log \left(\frac{\Omega_s}{\Omega_p} \right)}$$

$$N \geq \frac{\log \sqrt{\frac{10^{0.1 \times 15} - 1}{10^{0.1 \times 3} - 1}}}{\log \left(\frac{1000}{500} \right)} \Rightarrow N \geq 2.47$$

$\underline{N = 3}$

For HPF, $\Omega_c = \Omega_p = 1000$ rad/s & put $s = \frac{\Omega_c}{s}$

N=3

$$H(s) = \frac{1}{(s+1)(s^2+s+1)}$$

$$H_a(s) = \frac{1}{\left(\frac{1000}{s}+1\right)\left(\left(\frac{1000}{s}\right)^2 + \frac{1000}{s}+1\right)}$$

$$= \frac{s \cdot s^2}{(1000+s)(1000^2 + 1000s + s^2)}$$

$$= \frac{s^3}{10^9 + 10^6 s + 10^3 s^2 + 10^6 s + 10^3 s^2 + s^3}$$

$$H_a(s) = \frac{s^3}{s^3 + 2s^2 10^3 + 2s 10^6 + 10^9}$$

2. Design a digital butterworth filter that has -1dB passband attenuation at 200 Hz and atleast -15dB stopband attenuation at 540 Hz.

soln. $f_p = 200$ Hz ⇒ $\Omega_p = 2\pi f = 2\pi \times 200$

$\underline{\Omega_p = 400\pi}$

$f_s = 540$ Hz ⇒ $\Omega_s = 2\pi \times 540$ ⇒ $\underline{1080\pi} = \Omega_s$

$\alpha_p = -1dB$; $\alpha_s = -15dB$.

09/10/19 Wednesday

Design of IIR Filters from Analog Filters

For the conversion of analog filters into digital filters to be effective following conditions must be satisfied.

1. The $j\Omega$ axis in the s-plane should map into a unit circle in the z-plane.

2. Left half of s-plane should map into inside of the unit circle in the z-plane.

The following methods are used for the conversion of analog to digital filter.

1. Impulse invariance method.
2. Bilinear transformation method.

I

Impulse invariance method

In impulse invariance method, the IIR filter is designed such that the unit impulse response $h(n)$ of the digital filter is the sampled version of impulse response of analog filter. We know,

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n}$$

$$H(z)\Big|_{z=e^{sT}} = \sum_{n=0}^{\infty} h(n) e^{-sTn}$$

Let us consider mapping of points from s-plane to z-plane implied by the relation $z = e^{sT}$.

But, $s = \sigma + j\Omega$ & let $z = re^{j\omega}$

$$\Rightarrow re^{j\omega} = e^{(\sigma+j\Omega)T}$$

$$re^{j\omega} = e^{\sigma T} \cdot e^{j\Omega T}$$

On comparing, $r = e^{\sigma T}$ —— ①

$\omega = \Omega T$ —— ②

To check the 2 essential conditions for effective conversion from analog to digital filter, we first consider the mapping of $j\Omega$ axis. $j\Omega$ axis means $\sigma = 0$.

Put $\sigma = 0$ in equ. ①

$$r = e^{0} \Rightarrow r = 1$$

So $j\Omega$ axis maps into a unit circle.

To check the 2nd condition we consider the left half of s-plane. In the left half of s-plane, $\sigma < 0$.

So $e^{\sigma T}$ becomes $e^{(-ve)\text{ value}} < 1$

$$\Rightarrow \frac{1}{e^{\sigma T}} < 1$$

So the left half of s-plane maps into the interior of a unit circle.

* Steps to design a digital filter using impulse invariance method:

1. For the gn specifications find $H_a(s)$ (transfer fn of the analog filter)
2. Select the sampling rate of the digital filter - T s/sample
3. Express the analog filter transfer function in the following way:

$$H_a(s) = \sum_{k=1}^{N} \frac{C_k}{s-P_k}$$

$$= \frac{C_1}{s-P_1} + \frac{C_2}{s-P_2} + \frac{C_3}{s-P_3} + \cdots$$

4. Compute the z-transform of the digital filter by using the formula,

$$H(z) = \sum_{k=1}^{N} \frac{C_k}{1-e^{p_k T} z^{-1}}$$

$$= \frac{C_1}{1-e^{p_1 T} z^{-1}} + \frac{C_2}{1-e^{p_2 T} z^{-1}} + \frac{C_3}{1-e^{p_3 T} z^{-1}} + \dots$$

Q.
1. For the analog transfer function $H(s) = \frac{2}{(s+1)(s+2)}$, find $H(z)$ using impulse invariance method. Assume $T = 1$.

soln:

$$H(s) = \frac{2}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$\Rightarrow 2 = A(s+2) + B(s+1)$$

put $s = -2$,

$2 = -B \Rightarrow \underline{B = -2}$

put $s = -1$,

$\underline{2 = A}$

$$H(s) = \frac{2}{s-(-1)} + \frac{(-2)}{s-(-2)} \qquad \left[H(s) = \sum_{k=1}^{N} \frac{C_k}{s-p_k} \right]$$

$$H(z) = \frac{2}{1-e^{-1}z^{-1}} + \frac{(-2)}{1-e^{-2}z^{-1}} \qquad (T = 1\ s/\text{sample})$$

$$\left[H(z) = \sum_{k=1}^{N} \frac{C_k}{1-e^{-p_k T} z^{-1}} \right]$$

2. Design a third order butterworth digital filter using impulse invariant technique. Assume $T = 1$.

soln: $N = 3$.

$$H(s) = \frac{1}{(s+1)(s^2+s+1)} = \frac{A}{s+1} + \frac{B}{s^2+s+1}$$

~~$1 = A(s^2+s+1) + B(s+1)$~~

$$\frac{1}{(s+1)(s^2+s+1)} = \frac{A}{s+1} + \frac{B}{s-\left(\frac{-1}{2}+\frac{\sqrt{3}}{2}j\right)} + \frac{C}{s-\left(\frac{-1}{2}-\frac{\sqrt{3}}{2}j\right)}$$

$$1 = A(s^2+s+1) + B(s+1)\left(s-\left(\frac{-1}{2}-\frac{\sqrt{3}}{2}j\right)\right) + C(s+1)\left(s-\left(\frac{-1}{2}+\frac{\sqrt{3}}{2}j\right)\right)$$

Put $s = -1$,

$\underline{1 = A}$

Equating coeff.:

$s^2 \Rightarrow 0 = A + B + C \Rightarrow B + C = -1 \Rightarrow B = -C-1$

const $\Rightarrow 1 = A - \left(\frac{1}{2}+\frac{\sqrt{3}}{2}j\right)B - \left(\frac{1}{2}-\frac{\sqrt{3}}{2}j\right)C$

$$\Rightarrow \left(\frac{1}{2}+\frac{\sqrt{3}}{2}j\right)B + \left(\frac{1}{2}-\frac{\sqrt{3}}{2}j\right)C = 0$$

$$\left(\frac{1}{2}+\frac{\sqrt{3}}{2}j\right)(-C-1) + \left(\frac{1}{2}-\frac{\sqrt{3}}{2}j\right)C$$

$$C\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}j+\frac{1}{2}-\frac{\sqrt{3}}{2}j\right) = \frac{1}{2}+\frac{\sqrt{3}}{2}j \Rightarrow -\sqrt{3}j\,C = \frac{1}{2}+\frac{\sqrt{3}}{2}j$$

$C = \frac{1}{-\sqrt{3}j}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}j\right)$

$\Rightarrow C = -0.5 + 0.288j$

$\Rightarrow B = -0.5 - 0.288j$

$\therefore H(s) = \frac{1}{s+1} + \frac{(-0.5-0.288j)}{s-(0.5+0.866j)} + \frac{(-0.5+0.288j)}{s-(0.5-0.866j)}$

$= \frac{1}{s-(-1)} + \frac{(-0.5-0.288j)}{s-(0.5+0.866j)} + \frac{(-0.5+0.288j)}{s-(0.5-0.866j)}$

$\therefore H(z) = \frac{1}{1-e^{-1}z^{-1}} + \frac{(-0.5-0.288j)}{1-e^{-0.5+0.866j}z^{-1}} + \frac{(-0.5+0.288j)}{1-e^{-0.5-0.88j}z^{-1}}$

15/10/19 Tuesday

II Bilinear transformation method

The mapping of s-plane to z-plane in bilinear transformation is gn by,

$$S = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

We know,

$S = \sigma + j\Omega$ and $z = re^{j\omega}$

$$\sigma + j\Omega = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) = \frac{2}{T}\left(\frac{z-1}{z+1}\right)$$

$$= \frac{2}{T}\left(\frac{re^{j\omega}-1}{re^{j\omega}+1}\right)$$

$$= \frac{2}{T}\left(\frac{r(\cos\omega + j\sin\omega)-1}{r(\cos\omega + j\sin\omega)+1}\right)$$

$$= \frac{2}{T}\left(\frac{(r\cos\omega - 1) + j\sin\omega . r}{(r\cos\omega + 1) + jr\sin\omega}\right)$$

- Multiply & divide by the conjugate of denominator.
- Separate real & img parts

$$\sigma + j\Omega = \frac{2}{T}\left[\frac{r^2-1}{1+r^2+2r\cos\omega} + j\frac{2r\sin\omega}{1+r^2+2r\cos\omega}\right]$$

- Comparing,

$$\sigma = \frac{2}{T}\left(\frac{r^2-1}{1+r^2+2r\cos\omega}\right) \quad \text{—— (1)}$$

$$\Omega = \frac{2}{T}\left(\frac{2r\sin\omega}{1+r^2+2r\cos\omega}\right) \quad \text{—— (2)}$$

subst. $r = 1$ in (1),

$\sigma = 0$. $\Rightarrow j\Omega$ axis maps into a unit circle in z-plane.

Put $r < 1$ in (1)

$r^2 - 1 \approx -ve$

So we get a -ve value for $r^2-1 \Rightarrow \sigma < 0$. So the left half of s-plane is mapped into interior of unit circle in z-plane.

* Warping effect

put $r = 1$ in equ. ②,

$$\Omega = \frac{2}{T}\left(\frac{2\sin\omega}{1+1+2\cos\omega}\right)$$

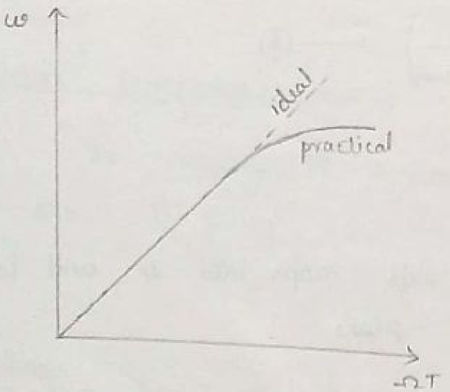
$$= \frac{2}{T}\left(\frac{2\sin\omega}{2(1+\cos\omega)}\right) = \frac{2}{T}\left(\frac{2\sin\omega/2 \cdot \cos\omega/2}{2\cos^2\omega/2}\right)$$

$$\Omega = \frac{2}{T}\tan\omega/2$$

For small values of ω, $\tan\omega/2 = \omega/2$. (if $\theta << , \tan\theta \approx \theta$)

$$\Omega = \frac{2}{T}\,\omega/2$$

$$\Omega = \frac{\omega}{T} \Rightarrow \omega = \Omega T$$



For low freq, the relation b/w ω & Ω are linear. But as the freq increase, the relationship is non-linear & a distortion is introduced in the freq state and this is called as Warping effect.

It is eliminated by pre-warping the analog filter.

Formula used for pre-warping is

$$\Omega = \frac{2}{T}\tan\omega/2.$$

* Steps to design the digital filter using bilinear transformation

1. From the gn specifications find the pre-warping analog freq.
2. Find the analog filter transfer fn using the freq obtained in step 1.
3. Transform the analog filter to a digital filter using the formula

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

Q.

1. Apply bilinear transformation on $H(s) = \frac{2}{(s+1)(s+2)}$ with $T = 1$

soln: ~~put s~~ $H(s) = \frac{2}{(s+1)\,(s+2)}$

put $s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$

$T = 1 \Rightarrow s = 2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$

$$H(s) = \frac{2}{\left(2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+1\right)\left(2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+2\right)}$$

$$= \frac{1}{\left(\frac{2(1-z^{-1})+(1+z^{-1})}{1+z^{-1}}\right)\left(\frac{1-z^{-1}+1+z^{-1}}{1+z^{-1}}\right)}$$

$$H(s) = \frac{(1+z^{-1})^2}{(3-z^{-1})\times 2}$$

2. Using bilinear transformation, design a high pass filter monotonic in pass band with cut-off freq of 1000Hz and down 10 dB at 350 Hz. The sampling freq is 5000 Hz. (down → stop band)

soln:

Monotonic in passband ⇒ butterworth filter.

$\alpha_s = 10\,dB$

$f_s = 350\,Hz$

$\omega_s = 2\pi \times 350 = 700\pi$ (ω_s - since bilinear)

$f_c = f_p = 1000\,Hz$

$\omega_c = \omega_p = 2\times\pi\times 1000 = 2000\pi$

Here, $\alpha_p = 3dB$.

Since, we have to design using bilinear transformation, pre-warping is to be applied.

$$\Omega = \frac{2}{T}\tan \omega/2$$

sampling freq, $F_S = 5000\,Hz$

$$T = \frac{1}{F_S} = \frac{1}{5000} = 2\times10^{-4}$$

∴ Here, $\Omega = \frac{2}{T}\tan\frac{\omega T}{2}$

$$\Omega_p = \frac{2}{T}\tan\frac{\omega_p T}{2} = \frac{2}{2\times10^{-4}}\tan\frac{2000\pi\times 2\times10^{-4}}{2}$$

$$= 7265.43$$

$$\Omega_s = \frac{2}{T}\tan\frac{\omega_s T}{2} = \frac{2}{2\times10^{-4}}\tan\frac{700\pi\times2\times10^{-4}}{2}$$

$$= 2235.26$$

Now,

$\alpha_s = 10\,dB$ $\quad \alpha_p = 3\,dB$

$\Omega_s = 2235$ $\quad \Omega_p = 7265$ ⇒ for HPF.

For LPF,

$\Omega_p = 2235$ $\quad \Omega_s = 7265$

$\alpha_p = 3\,dB$ $\quad \alpha_s = 10\,dB$.

$$N \geq \frac{\log\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$$N \geq \frac{\log\sqrt{\frac{10^{0.1\times10}-1}{10^{0.1\times3}-1}}}{\log\left(\frac{7265}{2235}\right)} \Rightarrow N \geq 0.93 \Rightarrow N = 1.$$

For HPF,

$\Omega_c = \Omega_p = 7265$

$N=1$ $\quad H(s) = \dfrac{1}{s+1}$

$s = \dfrac{\Omega_c}{s} = \dfrac{7265}{s}$

$\Rightarrow H(s) = \dfrac{1}{\frac{7265}{s}+1} = \dfrac{s}{7265+s}$

For bilinear transformation, $s = \dfrac{2}{T}\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)$

$$H(s) = \frac{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}{7265 + \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{\frac{2}{2\times10^{-4}}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}{7265 + \frac{2}{2\times10^{-4}}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{10^4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}{\frac{7265(1+z^{-1}) + 10^4(1-z^{-1})}{(1+z^{-1})}}$$

$$H(s) = \frac{10^4(1-z^{-1})}{17265 - 2735z^{-1}}$$

16/10/19 Wednesday

Frequency Transformation in Digital Domain

LPF → LPF

$z^{-1} \rightarrow \dfrac{z^{-1}-\alpha}{1-\alpha z^{-1}}$ → ω_p' = passband freq of new LPF

$\alpha = \dfrac{\sin[(\omega_p - \omega_p')/2]}{\sin[(\omega_p + \omega_p')/2]}$ → ω_p = passband freq. of LPF

LPF → HPF

$z^{-1} \rightarrow -\left[\dfrac{z^{-1}+\alpha}{1+\alpha z^{-1}}\right]$ → ω_p' = passband freq of HPF

$\alpha = \dfrac{-\cos[(\omega_p' + \omega_p)/2]}{\cos[(\omega_p' - \omega_p)/2]}$

LPF → BPF

$$z^{-1} \rightarrow -\frac{\left(z^{-2} - \frac{2\alpha K}{1+K}z^{-1} + \frac{K-1}{K+1}\right)}{\frac{K-1}{K+1}z^{-2} - \frac{2\alpha K}{K+1}z^{-1} + 1}$$

→ $K = \cot\left(\dfrac{\omega_u - \omega_l}{2}\right)\tan\dfrac{\omega_p}{2}$ (ω_u → upper cut-off freq, ω_l → lower cut-off freq)

$\alpha = \dfrac{\cos[(\omega_u + \omega_l)/2]}{\cos[(\omega_u - \omega_l)/2]}$

LPF → BSF

$$z^{-1} \rightarrow \frac{z^{-2} - \frac{2\alpha}{1+K}z^{-1} + \frac{1-K}{1+K}}{\frac{1-K}{1+K}z^{-2} - \frac{2\alpha}{1+K}z^{-1} + 1}$$

→ $K = \tan((\omega_u - \omega_l)/2) \cdot \tan\dfrac{\omega_p}{2}$

UQ *

1. Design a digital butterworth LPF with $\omega_p = \pi/6$, $\omega_s = \pi/4$, min passband gain = −2 dB and min stop-band attenuation = 8 dB. Use bilinear transformation. Take T = 1.

soln:

$\alpha_p = 2\,dB \qquad \alpha_s = 8\,dB.$

$\omega_p = \pi/6 \qquad \omega_s = \pi/4.$

We have, (pre-warping) $\Omega = \frac{2}{T}\tan\omega/2 \qquad T = 1.$

$\Omega_p = \frac{2}{T}\tan\omega_p/2 = 2\tan\frac{\pi}{12}$

$= 0.536$

$\Omega_s = \frac{2}{T}\tan\frac{\omega_s}{2} = 2\tan\frac{\pi}{8}$

$= 0.828$

$$N \geq \frac{\log\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log\left(\frac{\Omega_s}{\Omega_p}\right)} \Rightarrow N \geq \frac{\log\sqrt{\frac{10^{0.1\times 8}-1}{10^{0.1\times 2}-1}}}{\log\left(\frac{0.828}{0.536}\right)}$$

$N \geq 2.54 \Rightarrow \underline{N = 3}$

$$\Omega_c = \frac{\Omega_p}{(10^{0.1\alpha_p}-1)^{1/2N}} = \frac{0.536}{(10^{0.1\times 2}-1)^{\frac{1}{2\times 3}}} = \underline{0.586}$$

N = 3 $\quad H(s) = \frac{1}{(s+1)(s^2+s+1)}$

put $s = \frac{s}{\Omega_c} = \frac{s}{0.586}$

$$\Rightarrow H_a(s) = \frac{1}{\left(\frac{s}{0.586}+1\right)\left(\left(\frac{s}{0.586}\right)^2 + \frac{s}{0.586}+1\right)}$$

$$= \frac{0.586 \times 0.586^2}{(s+0.586)(s^2+0.586s+0.586^2)}$$

$$H_a(s) = \frac{0.201}{(s+0.586)(s^2+0.586s+0.343)}$$

2. Design a butterworth filter using bilinear transformation.

$0.707 \leq |H(e^{j\omega})| \leq 1$

By bilinear transformation, $s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$

$s = 2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$

$$H(s) = \frac{0.201}{\left(2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+0.586\right)\left(4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 0.586\times 2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+0.343\right)}$$

$$= \frac{0.201}{\left(\frac{2(z-1) + 0.586(z+1)}{z+1}\right)\left(\frac{4(z-1)^2 + 1.172(z-1)(z+1) + 0.343(z+1)^2}{(z+1)^2}\right)}$$

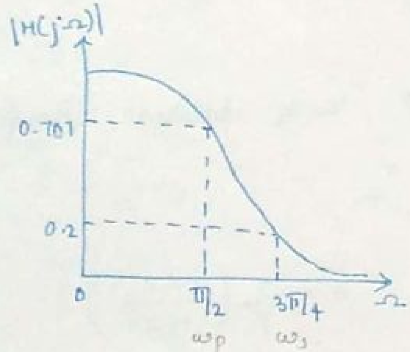
$$H(s) = \frac{0.201(z+1)^3}{(2.586z - 1.414)\left(4(z-1)^2 + 1.172(z^2-1) + 0.343(z+1)^2\right)}$$

2. Design butterworth filter using bilinear transformation.

$0.707 \le |H(e^{j\omega})| \le 1$ for $0 \le \omega \le \pi/2$

$|H(e^{j\omega})| \le 0.2$ for $3\pi/4 < \omega < \pi$; $T = 1$

soln:



$$\frac{1}{\sqrt{1+\varepsilon^2}} = 0.707 \Rightarrow 1+\varepsilon^2 = \left(\frac{1}{0.707}\right)^2$$

$$\varepsilon^2 = \frac{1}{0.707^2} - 1 \Rightarrow \varepsilon = \sqrt{1.0006} = 1.0002$$

$$\varepsilon \approx 1$$

$$\frac{1}{\sqrt{1+\lambda^2}} = 0.2 \Rightarrow \lambda^2 = \frac{1}{0.2^2} - 1$$

$$\lambda = \sqrt{24} = 4.898$$

$$\Omega = \frac{2}{T}\tan\omega/2 \quad ; T = 1$$

$$\Omega_p = 2\tan\omega_p/2 = 2\tan(\pi/4) = 2$$

$$\Omega_s = 2\tan\omega_s/2 = 2\tan(3\pi/8) = 4.828$$

$$N \ge \frac{\log\left(\frac{\lambda}{\varepsilon}\right)}{\log\left(\frac{\Omega_s}{\Omega_p}\right)} \Rightarrow N \ge \frac{\log\left(\frac{4.898}{1}\right)}{\log\left(\frac{4.828}{2}\right)}$$

$$N \ge 1.80 \Rightarrow \underline{N = 2}$$

$$\Omega_c = \frac{\Omega_p}{\varepsilon^{1/2N}} = \frac{2}{1^{1/2\times2}} = \underline{2}$$

⋮

3. Why can't we use impulse invariance technique for implementing digital HPF.?

Ans: The $j\Omega$ axis in s-plane is mapped into unit circle in z-plane. But this mapping is not one-to-one mapping. But it is many-to-one mapping. It means that many points in s-plane are mapped to a single point in the z-plane.

Consider 2 poles in s-plane with identical real parts but with img components differ in by $2\pi/T$. So let the poles be $s_1 = \sigma + j\Omega$

$$s_2 = \sigma + j\left(\Omega + 2\pi/T\right)$$

Now, let these poles be mapped into z-planes using impulse invariance mapping. $z_1 = e^{s_1 T}$ & $z_2 = e^{s_2 T}$

$$z_1 = e^{s_1 T} = e^{(\sigma + j\Omega)T}$$

$$z_2 = e^{s_2 T} = e^{(\sigma + j(\Omega + 2\pi/T))T}$$

$$= e^{(\sigma + j\Omega)T} \cdot e^{j2\pi}$$

But we know that $e^{j2\pi} = 1$, So $z_2 = e^{(\sigma + j\Omega)T}$

So $z_1 = z_2$.

So we find that these two poles map to same location in z-plane. So there are, infinite no. of s-plane poles that map into the same locations in z-plane. They must have same real parts & img parts that differ by some integer multiple ~~by~~ of $\frac{2\pi}{T}$.

The main disadv. of impulse invariance mapping is aliasing caused by s-plane poles having ~~imp~~ img. parts differing by integral multiple of $2\pi/T$. Due to the presence of aliasing we use impulse invariance method for LPF & BPF, only. It is not used for HPF.

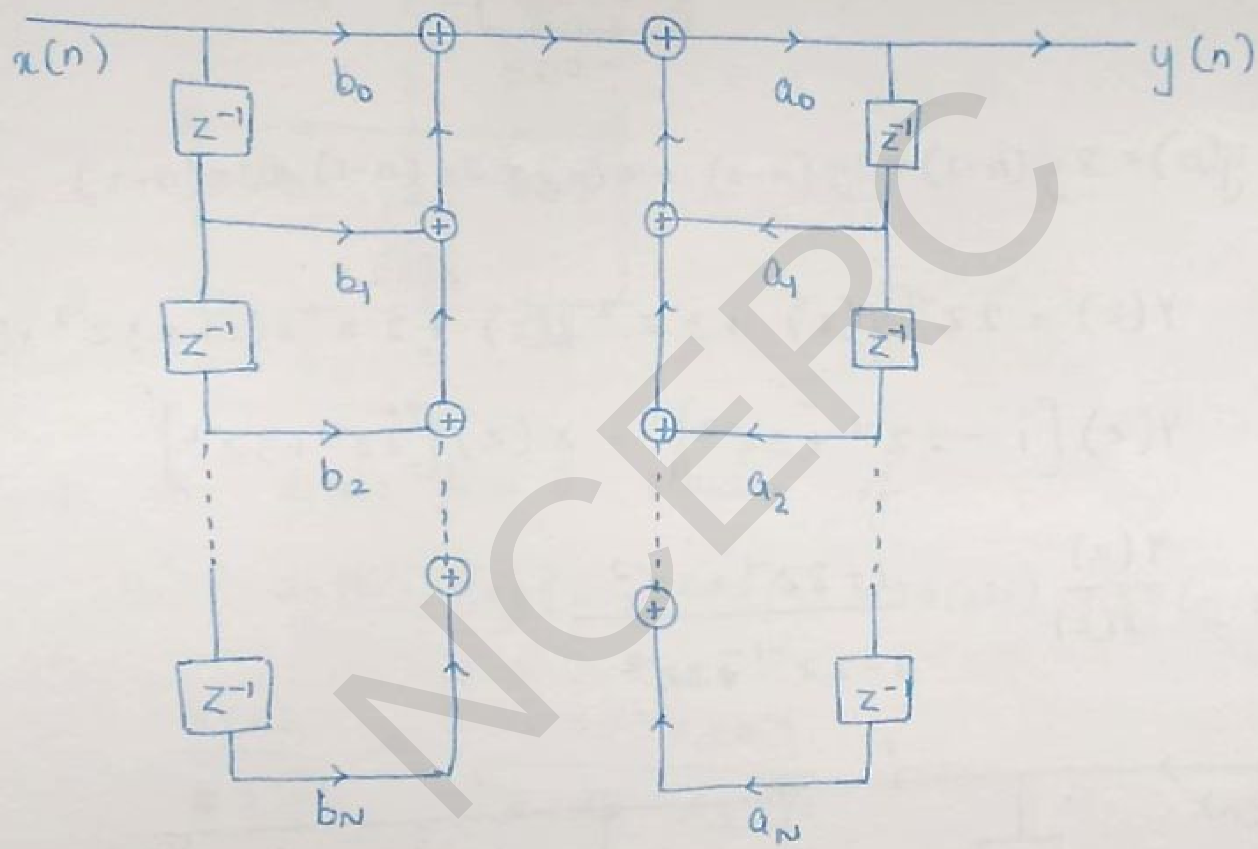
22/10/19 Tuesday.

MODULE - 5

STRUCTURES FOR REALISATION OF DIGITAL FILTERS

IIR filters

1. Direct - form I (DF - I)



Q.

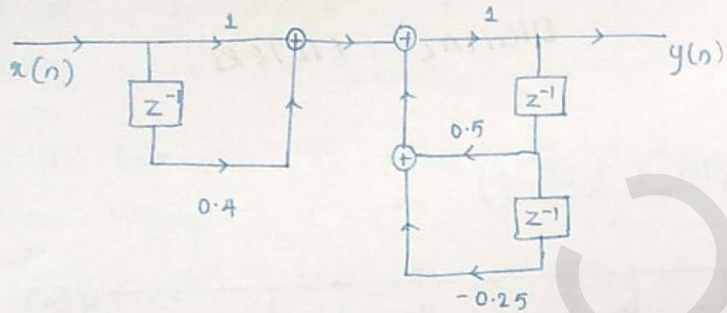
1. Obtain the DF - I realisation of the s/m described by the difference equ. : $y(n) = 0.5\,y(n-1) - 0.25\,y(n-2) + x(n) + 0.4\,x(n-1)$

soln : Take z- transform on both sides of the difference equ.

$$Y(z) = 0.5\,z^{-1}Y(z) - 0.25\,z^{-2}Y(z) + X(z) + 0.4\,z^{-1}X(z)$$

$$Y(z)\left[1 - 0.5z^{-1} + 0.25z^{-2}\right] = X(z) + 0.4z^{-1}X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{1 + 0.4 z^{-1}}{1 - 0.5 z^{-1} + 0.25 z^{-2}}$$



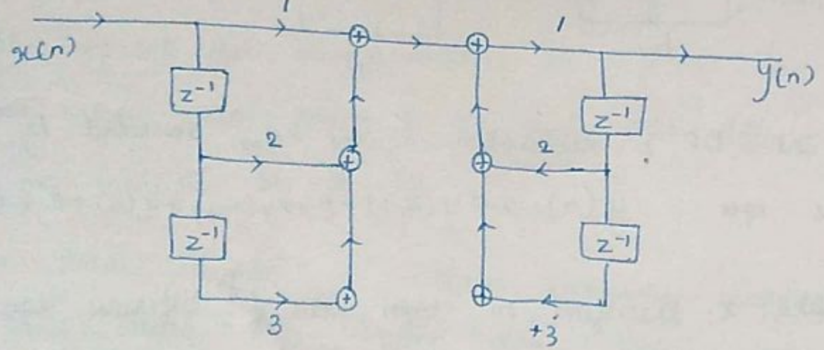
2. $y(n) = 2y(n-1) + 3y(n-2) + x(n) + 2x(n-1) + 3x(n-2)$

soln :

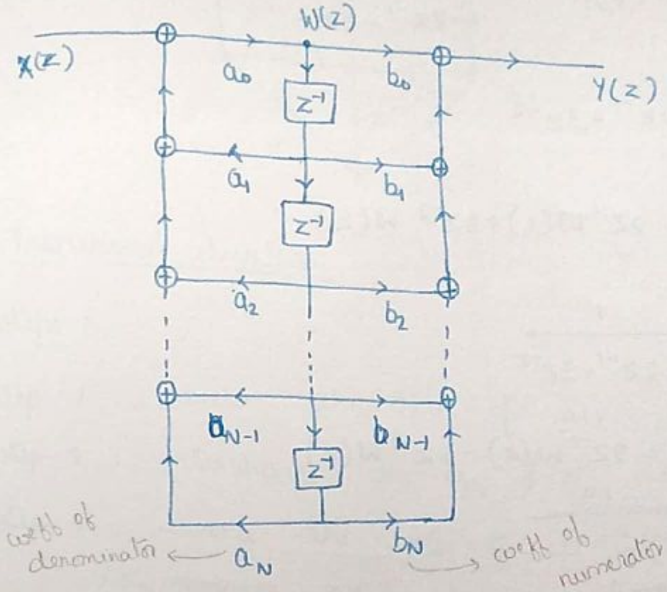
$$Y(z) = 2z^{-1}Y(z) + 3z^{-2}Y(z) + X(z) + 2z^{-1}X(z) + 3z^{-2}X(z)$$

$$Y(z)\left[1 - 2z^{-1} - 3z^{-2}\right] = X(z)\left[1 + 2z^{-1} + 3z^{-2}\right]$$

$$\frac{Y(z)}{X(z)} = \frac{1 + 2z^{-1} + 3z^{-2}}{1 - 2z^{-1} - 3z^{-2}}$$



2. Direct form II (DF - II)

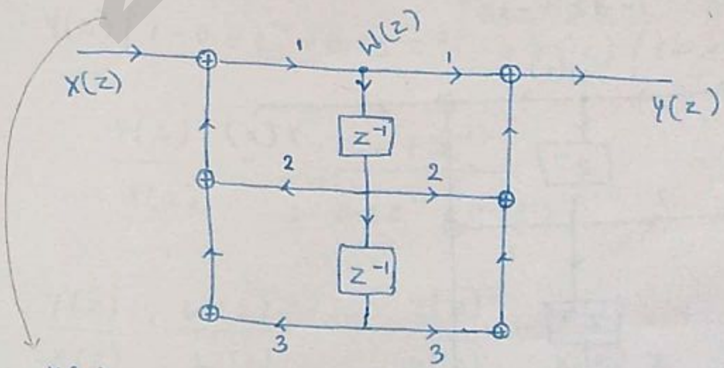


Q1. Find the direct form II structure of

$$y(n) = 2y(n-1) + 3y(n-2) + x(n) + 2x(n-1) + 3x(n-2)$$

soln :

$$\frac{Y(z)}{X(z)} = \frac{1 + 2z^{-1} + 3z^{-2}}{1 - 2z^{-1} - 3z^{-2}}$$



$$\frac{Y(z)}{X(z)} = \frac{Y(z)}{X(z)} \cdot \frac{W(z)}{W(z)}$$

$$= \frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)} = \frac{1+2z^{-1}+3z^{-2}}{1-2z^{-1}-3z^{-2}}$$

$$\frac{Y(z)}{W(z)} = 1+2z^{-1}+3z^{-2}$$

$$\Rightarrow Y(z) = W(z) + 2z^{-1}W(z) + 3z^{-2}W(z)$$

$$\& \frac{W(z)}{X(z)} = \frac{1}{1-2z^{-1}-3z^{-2}}$$

$$\Rightarrow X(z) = W(z) - 2z^{-1}W(z) - 3z^{-2}W(z)$$

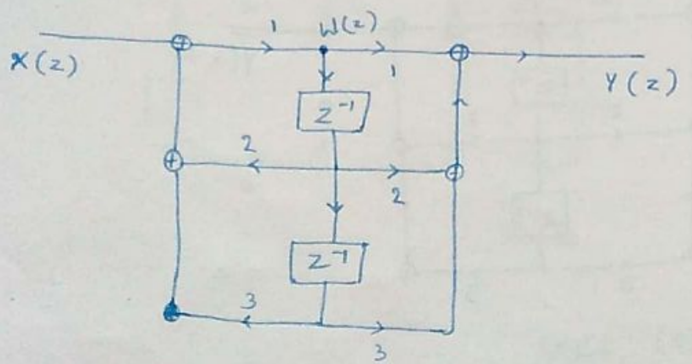
3. Signal flow graph

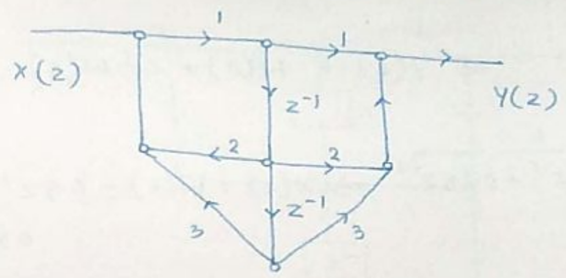
Q.1. Draw the signal flow graph of following s/m given by difference equ:

$$y(n) = 2y(n-1) + 3y(n-2) + x(n) + 2(x(n-1)) + 3x(n-2)$$

soln:

$$\frac{Y(z)}{X(z)} = \frac{1+2z^{-1}+3z^{-2}}{1-2z^{-1}-3z^{-2}}$$





4 Transposed structure

steps :

step 1 : reverse the dir. of all the branches.

step 2 : interchange the i/ps & o/ps

step 3 : reverse the roles of all the nodes in branches. i.e., summing points becomes branching points & vice-versa.

Q. 1. Determine the DF-II and transposed DF-II for the s/m $y(n) = \frac{1}{2}y(n-1) - \frac{1}{4}y(n-2) + x(n) + x(n-1)$

soln:

$$Y(z) = \frac{1}{2}z^{-1}Y(z) - \frac{1}{4}z^{-2}Y(z) + X(z) + z^{-1}X(z)$$

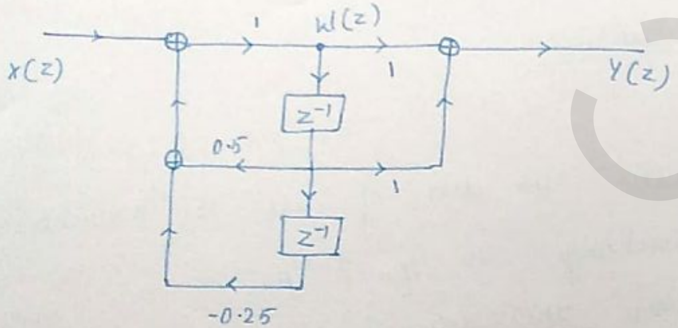
$$Y(z)\left[1-0.5z^{-1}+0.25z^{-2}\right] = X(z)\left[1+z^{-1}\right]$$

$$\frac{Y(z)}{X(z)} = \frac{1+z^{-1}}{1-0.5z^{-1}+0.25z^{-1}}$$

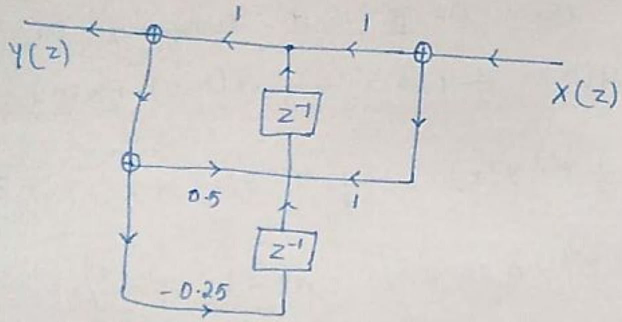
$$\frac{Y(z)}{X(z)} \cdot \frac{W(z)}{W(z)} = \frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)} = \frac{1+z^{-1}}{1-0.5z^{-1}+0.25z^{-2}}$$

$$\frac{Y(z)}{W(z)} = 1 + z^{-1} \Rightarrow Y(z) = W(z) + z^{-1} W(z)$$

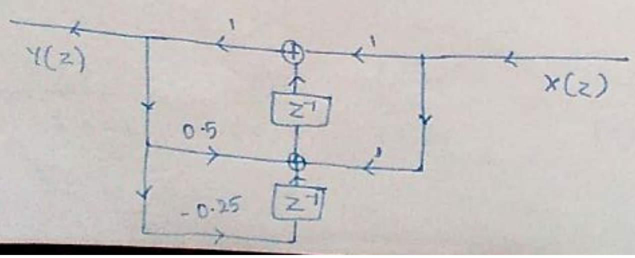
$$\frac{W(z)}{X(z)} = \frac{1}{1 - 0.5z^{-1} + 0.25z^{-2}} \Rightarrow X(z) = W(z) - 0.5z^{-1} W(z) + 0.25z^{-2} W(z)$$



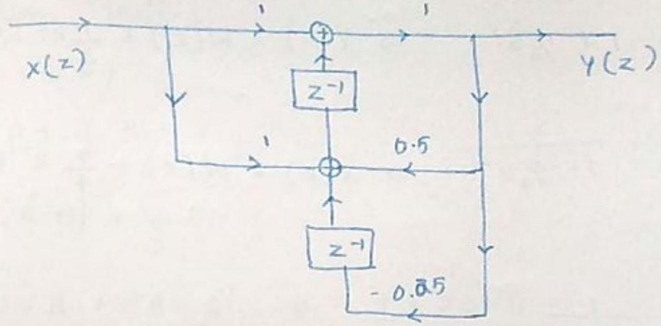
reverse the dir. of branches & interchange i/p & o/p.



reverse the roles of all nodes in branches.



mirror image



5. <u>Cascade.</u>

Q. Realise the s/m gn below in cascade form.

soln:
$$H(z) = \frac{1 + 1/3\, z^{-1}}{1 - 3/4\, z^{-1} + 1/8\, z^{-2}}$$

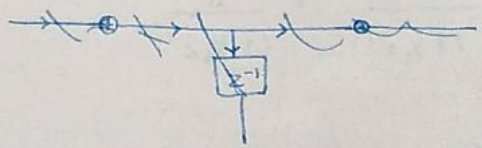
soln:

$$\left[1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right] \frac{z^2}{z^2}$$

$$\frac{z^2 - \frac{3}{4}z + \frac{1}{8}}{z^2} = \frac{(z - 1/4)}{z} \cdot \frac{(z - 1/2)}{z}$$

$$= \left(1 - \frac{1}{4}z^{-1}\right)\left(1 - 1/2\, z^{-1}\right)$$

$$H(z) = \frac{1 + 3/4\, z^{-1}}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)} = \left[\frac{1 + \frac{3}{4}z^{-1}}{1 - \frac{1}{4}z^{-1}}\right]\left[\frac{1}{1 - \frac{1}{2}z^{-1}}\right]$$



① $\frac{Y(z)}{X(z)} = \frac{1 + 3/4\, z^{-1}}{1 - 1/4\, z^{-1}}$

$$\frac{Y(z)}{X(z)} \cdot \frac{W(z)}{W(z)} = \frac{1 + 3/4\, z^{-1}}{1 - 1/4\, z^{-1}}$$

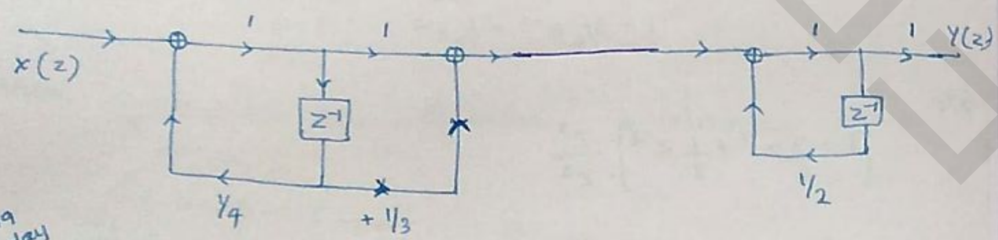
$$\frac{Y(z)}{W(z)} = 1 + \tfrac{1}{3}z^{-1} \Rightarrow Y(z) = W(z) + \frac{1}{3}z^{-1}W(z)$$

$$\frac{W(z)}{X(z)} = \frac{1}{1 - \frac{1}{4}z^{-1}} \Rightarrow X(z) = W(z) - \frac{1}{4}z^{-1}W(z)$$

② $$\frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{1}{2}z^{-1}}$$

$$Y(z) = W(z)$$

$$X(z) = W(z) - \frac{1}{2}z^{-1}W(z)$$



29/10/19 Tuesday

6. **Parallel form.**

Q. Realise the following s/m in parallel form.

$$H(z) = \frac{1 + z^{-1} + z^{-2}}{(1 + \frac{1}{2}z^{-1})(1 + \frac{1}{6}z^{-1})}$$

soln:

$$\frac{1 + z^{-1} + z^{-2}}{(1 + \frac{1}{2}z^{-1})(1 + \frac{1}{6}z^{-1})} = \frac{A}{1 + \frac{1}{2}z^{-1}} + \frac{B}{1 + \frac{1}{6}z^{-1}}$$

$$1 + z^{-1} + z^{-2} = A\left(1 + \frac{1}{6}z^{-1}\right) + B\left(1 + \frac{1}{2}z^{-1}\right)$$

Equating coeff.,

$$A + B = 1$$

$$\frac{1}{6}A + \frac{1}{2}B = 1$$

$$\frac{1}{6}(1 - B) + \frac{1}{2}B = 1$$

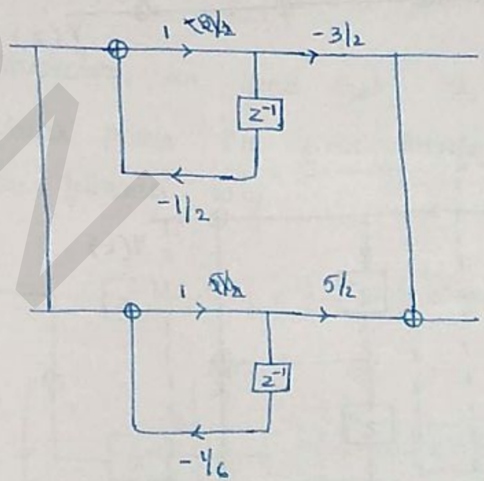
$$1 - B + 3B = 6 \Rightarrow 1 + 2B = 6$$

$$B = 5/2$$

$$A = 1 - 5/2$$

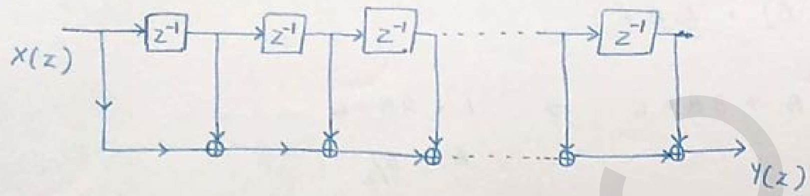
$$A = -3/2$$

$$H(z) = \frac{-3/2}{1 + \frac{1}{2}z^{-1}} + \frac{5/2}{1 + \frac{1}{6}z^{-1}}$$



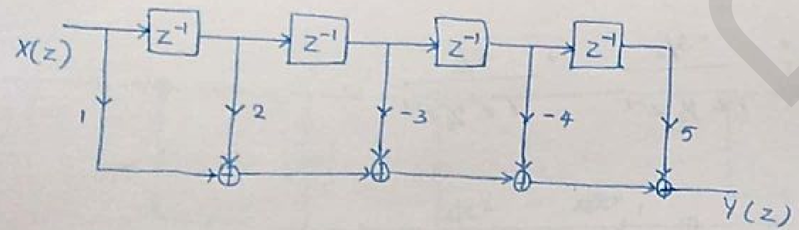
FIR filters

1. Direct form.

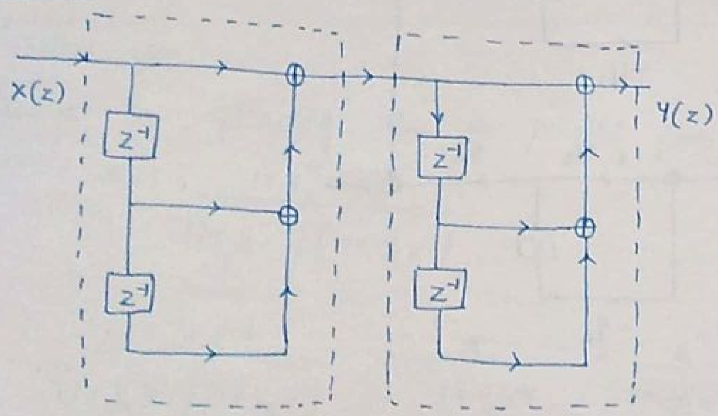


Q. Realise following filter using direct form $1+2z^{-1}-3z^{-2}-4z^{-3}+5z^{-4}$

soln:

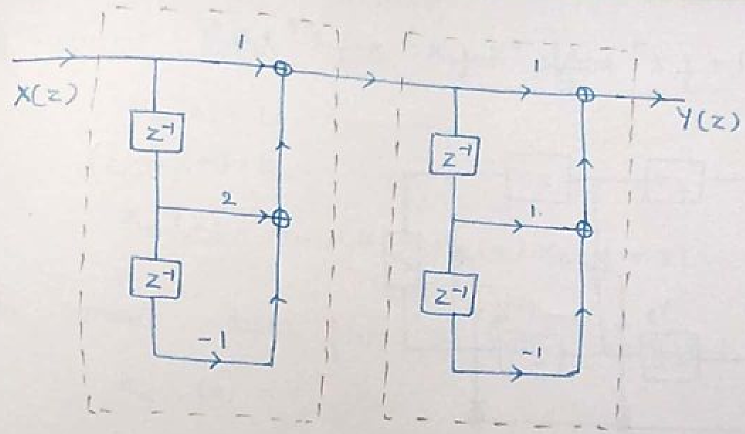


2. Cascade



Q. Realise the following filter in cascade form.

$$(1+2z^{-1}-z^{-2})(1+z^{-1}-z^{-2})$$

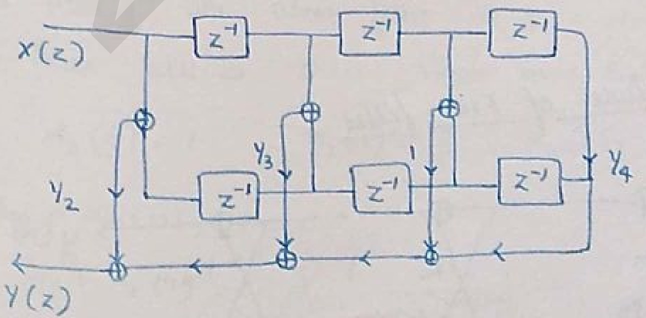


Realisation of linear phase FIR filters

(Realisation with minimum number of multipliers)

Q.1. Realise the following s/m: $H(z) = \frac{1}{2}+\frac{1}{3}z^{-1}+z^{-2}+\frac{1}{4}z^{-3}+z^{-4}+\frac{1}{3}z^{-5}+\frac{1}{2}z^{-6}$.

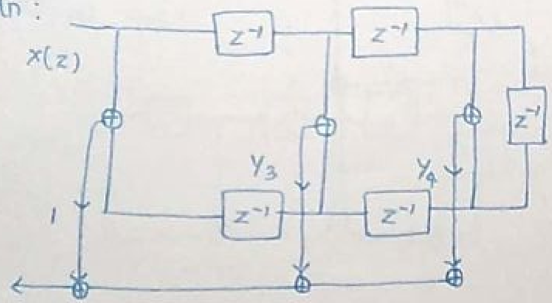
soln: By inspection, we find that the s/m function is that of linear phase FIR filter. Therefore, we realise the s/m in the following way.



(even)

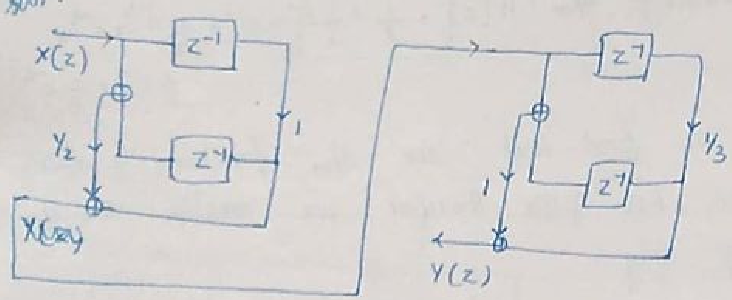
2. $H(z) = 1 + \frac{1}{3}z^{-1} + \frac{1}{4}z^{-2} + \frac{1}{4}z^{-3} + \frac{1}{3}z^{-4} + z^{-5}$

soln : (odd)



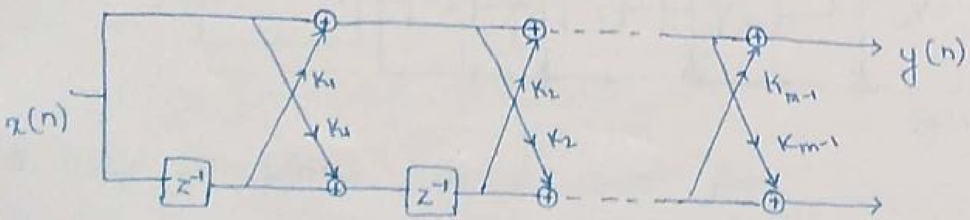
3. $H(z) = \left(\frac{1}{2} + z^{-1} + \frac{1}{2}z^{-2}\right)\left(1 + \frac{1}{3}z^{-1} + z^{-2}\right)$

soln:



30/10/19 Wednesday

<u>Lattice structure of FIR filter</u>



To convert lattice to direct form

$\alpha_m(0) = 1$

$\alpha_m(m) = K_m$

$\alpha_m(K) = \alpha_{m-1}(K) + \alpha_m(m)\, \alpha_{m-1}(m-K)$

To convert direct form to lattice coefficients

$\alpha_{m-1}(0) = 1$

$K_m = \alpha_m(m)$

$$\alpha_{m-1}(K) = \frac{\alpha_m(K) - \alpha_m(m) \cdot \alpha_m(m-K)}{1 - \alpha_m^2(m)}$$

Q.1. Find the lattice structure of

$$y(n) = 2x(n) + \frac{4}{5}x(n-1) + \frac{3}{2}x(n-2) + \frac{2}{3}x(n-3)$$

soln:

Coeff. of $x(n)$ should always be 1.

$$y(n) = 2\left[x(n) + \frac{2}{5}x(n-1) + \frac{3}{4}x(n-2) + \frac{1}{3}x(n-3)\right]$$

The order of the above filter is 3 → $x(n-3)$

So there will be three stages in lattice structure.

$\alpha_3(0) = 1$; $\alpha_3(1) = \frac{2}{5}$; $\alpha_3(2) = \frac{3}{4}$; $\alpha_3(3) = \frac{1}{3}$.

$\therefore K_m = \alpha_m(m)$,

$K_1 = \alpha_1(1)$;

$K_2 = \alpha_2(2)$

$K_3 = \alpha_3(3) = \frac{1}{3}$

$$\alpha_{m-1}(K) = \frac{\alpha_m(K) - \alpha_m(m)\cdot\alpha_m(m-K)}{1-\alpha_m^2(m)}$$

$$\alpha_{3-1}(2) = \alpha_2(2) = \frac{\alpha_3(2) - \alpha_3(3)\cdot\alpha_3(3-2)}{1-\alpha_3^2(3)}$$

$$= \frac{3/4 - 1/3 \cdot 2/5}{1-(1/3)^2}$$

$$= \frac{\frac{3}{4} - \frac{2}{15}}{\frac{9-1}{9}}$$

$$\alpha_2(2) = \frac{111}{160} = 0.693\text{75}$$

$$\alpha_1(1) = \alpha_{2-1}(1) = \frac{\alpha_2(1) - \alpha_2(2)\cdot\alpha_2(1)}{1-\alpha_2^2(2)}$$

$m=2$
$k=1$

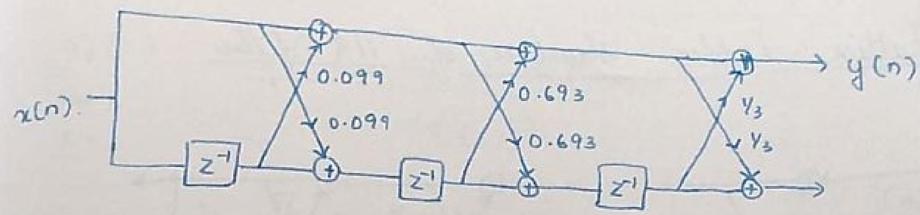
$\alpha_2(1) = ?$

$$\alpha_2(1) = \frac{\alpha_3(1) - \alpha_3(3)\cdot\alpha_3(2)}{1-(\alpha_3(3))^2}$$

$m=3$
$k=1$

$$= \frac{2/5 - 1/3 \cdot 3/4}{1-(1/3)^2} = \frac{27}{160} = 0.16875$$

$$\Rightarrow \alpha_1(1) = \frac{\frac{27}{160} - \frac{111}{160}\cdot\frac{27}{160}}{1-\left(\frac{111}{160}\right)^2} = 0.099$$



2. Consider the lattice coeff. $K_1 = 1/3$, $K_2 = 1/3$, $K_3 = 1/4$. Find the direct form coeff.

soln:

$\alpha_m(m) = K_m$

$\alpha_m(K) = \alpha_{m-1}(K) + \alpha_m(m)\,\alpha_{m-1}(m-K)$

We have to find $\alpha_3(0), \alpha_3(1), \alpha_3(2), \alpha_3(3)$ [∵ 3rd order]

$\alpha_3(3) = K_3 = 1/4$

$\alpha_3(2) = \alpha_2(2) + \alpha_3(3)\cdot\alpha_2(1)$

$\alpha_2(1) = \alpha_1(1) + \alpha_2(2)\cdot\alpha_1(1)$

$= 1/3 + 1/3 \cdot \frac{1}{3} = \frac{4}{9}$

$\Rightarrow \alpha_3(2) = \frac{1}{3} + \frac{1}{4}\cdot\frac{4}{9} = \frac{4}{9}$

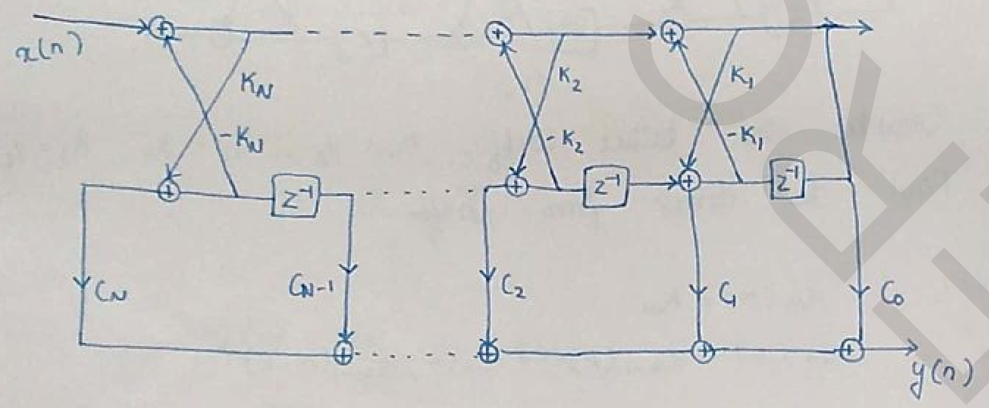
$\alpha_3(1) = \alpha_2(1) + \alpha_3(3)\cdot\alpha_2(2)$

$= \frac{4}{9} + \frac{1}{4}\cdot\frac{1}{3} = 0.527$

$\alpha_3(0) = 1$

$\therefore\ y(n) = x(n) + 0.527\,x(n-1) + \frac{4}{9}x(n-2) + \frac{1}{4}x(n-3)$

Lattice - Ladder structure of IIR filters



$$K_m = \alpha_m(m)$$

$$\alpha_{m-1}(k) = \frac{\alpha_m(k) - \alpha_m(m)\cdot\alpha_m(m-k)}{1 - \alpha_m^2(m)}$$

$$C_m = b_m - \sum_{i=m+1}^{M} C_i\,\alpha_i(i-m)$$

Q1. $H(z) = \dfrac{1 + 2z^{-1} + 2z^{-2} + z^{-3}}{1 + \frac{13}{24}z^{-1} + \frac{5}{8}z^{-2} + \frac{1}{3}z^{-3}}$

b_m - coeff. of numerator

$\alpha_m(-)$ - coeff of denominator

soln:

$b_0 = 1$ $\quad b_2 = 2$

$b_1 = 2$ $\quad b_3 = 1$

$\alpha_3(0) = 1 \qquad \alpha_3(2) = 5/8$

$\alpha_3(1) = 13/24 \qquad \alpha_3(3) = 1/3$

$K_1 = \alpha_1(1)\ ;\ K_2 = \alpha_2(2)\ ;\ K_3 = \alpha_3(3) = \frac{1}{3}$

$$\alpha_2(2) = \frac{\alpha_3(2) - \alpha_3(3)\cdot\alpha_3(1)}{1 - (\alpha_3(3))^2}$$

$m = 3$, $k = 2$

$$= \frac{5/8 - 1/3 \cdot \frac{13}{24}}{1 - (1/3)^2} = \frac{1}{2}$$

$$\alpha_1(1) = \frac{\alpha_2(1) - \alpha_2(2)\cdot\alpha_2(1)}{1 - \alpha_2^2(2)}$$

$m = 2$, $k = 1$

$$\alpha_2(1) = \frac{\alpha_3(1) - \alpha_3(3)\cdot\alpha_3(2)}{1 - \alpha_3^3(3)}$$

$m = 3$, $k = 1$

$$= \frac{13/24 - 1/3 \cdot 5/8}{1 - (1/3)^2} = \frac{3}{8}$$

$$\Rightarrow \alpha_1(1) = \frac{3/8 - 1/2 \cdot 3/8}{1 - (1/2)^2} = 1/4$$

$K_1 = 1/4$

$K_2 = 1/2$

$K_3 = 1/3$

$$C_m = b_m - \sum_{i=m+1}^{M} C_i \alpha_i (i-m)$$

M = order

M = 3

m = 0, 1, 2, 3

$$C_3 = b_3 - \sum_{i=4}^{3} C_i \alpha_i (i-3)$$

0

$$\Rightarrow C_3 = b_3 = 1$$

$$C_2 = b_2 - \sum_{i=3}^{3} C_i \alpha_i (i-2)$$

$$= 2 - C_3 \alpha_3 (1) = 2 - 1 \cdot \frac{13}{24}$$

$$\Rightarrow C_2 = 1.458$$

$$C_1 = b_1 - \sum_{i=2}^{3} C_i \alpha_i (i-1)$$

$$= 2 - \left(C_2 \alpha_2 (1) + C_3 \alpha_3 (2)\right)$$

$$= 2 - \left(1.458 \times \frac{3}{8} + 1 \cdot \frac{5}{8}\right)$$

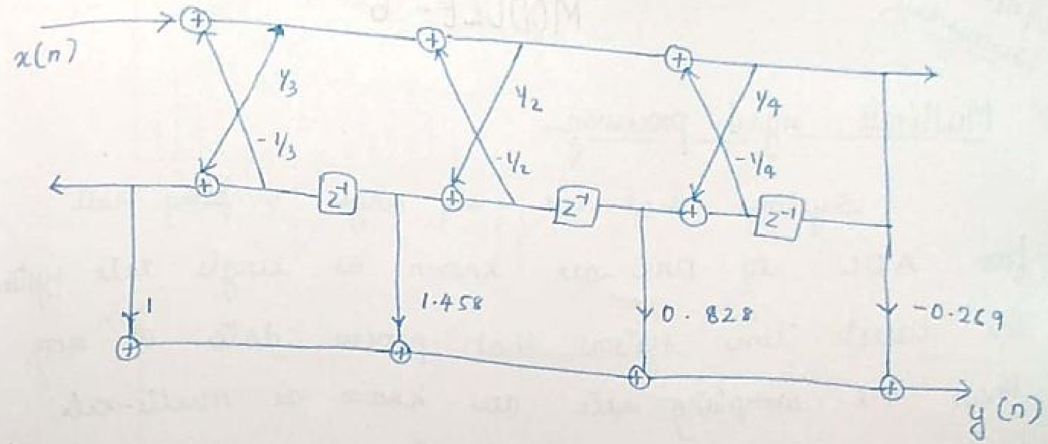
$$= 0.828$$

$$C_0 = b_0 - \sum_{i=1}^{3} C_i \alpha_i (i-0)$$

$$= 1 - \left(C_1 \alpha_1 (1) + C_2 \alpha_2 (2) + C_3 \alpha_3 (3)\right)$$

$$= 1 - \left(0.828 \times \frac{1}{4} + 1.458 \times \frac{1}{2} + 1 \times \frac{1}{3}\right)$$

$$= -0.269$$



IMP

* TMS 320C6TX DSP processor

- architecture
- block diagram
- explanation

6/11/19
Wednesday

MODULE - 6

Multirate signal processing

Systems that use single sampling rate from ADC to DAC are known as single rate systems. The discrete time systems that process data at more than one sampling rate are known as multi-rate systems.

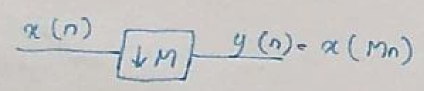
eg: (1) In video processing PAL and NTSC run at diff. sampling rates. So to watch an American programme in Europe, we need a sampling rate converter.

(2) In audio signal processing CD is sampled at 44.1 kHz but digital audio tape is sampled at 48 kHz. So here we need multirate signal processing techniques.

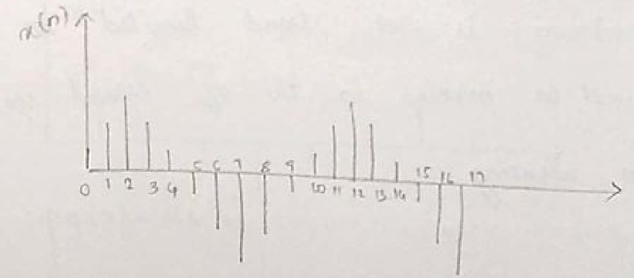
Down sampling (Decimator)

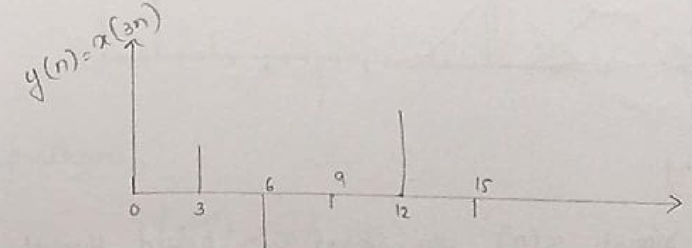
The sampling rate of a discrete time signal $x(n)$ can be reduced by a factor 'M' by taking every M^{th} value of the signal. It is represented by,

$$y(n) = x(Mn)$$



Block diagram.





Spectrum of the down sampled signal.

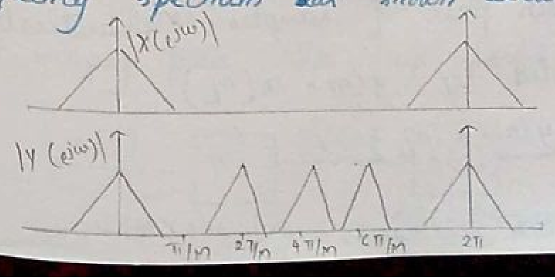
Let us consider an i/p signal $x(n)$. Let it be down sampled to get the o/p signal $y(n) = x(Mn)$.

Fourier transform of $y(n)$,

$$Y(e^{j\omega}) = \frac{1}{M} \sum_{k=0}^{M-1} X\left[e^{j(\omega - 2\pi k)/M}\right]$$

We find that fourier transform of the o/p signal is the sum of M uniformly shifted and stretched versions of $X(e^{j\omega})$ scaled by a factor $\frac{1}{M}$.

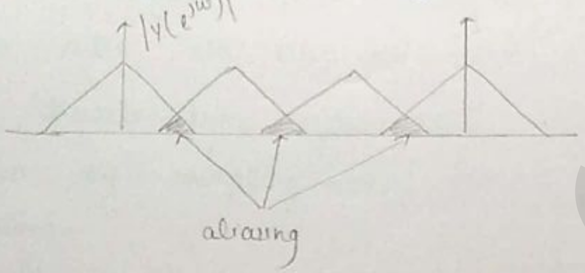
Frequency spectrum is as shown below.



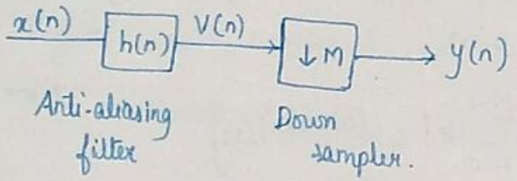
← spectrum of i/p

← spectrum of o/p

If the original spectrum is not band limited to $\omega = \pi/M$, there will be overlap in the o/p signal spectrum. This overlap causes aliasing.

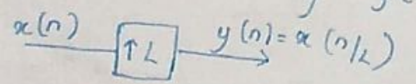


To bandlimit the signal $x(n)$, the signal is passed through LPF with cut-off freq π/M before down sampling. This filter is known as anti-aliasing filter. The complete process i.e., anti-aliasing & then down sampling is known as decimation.

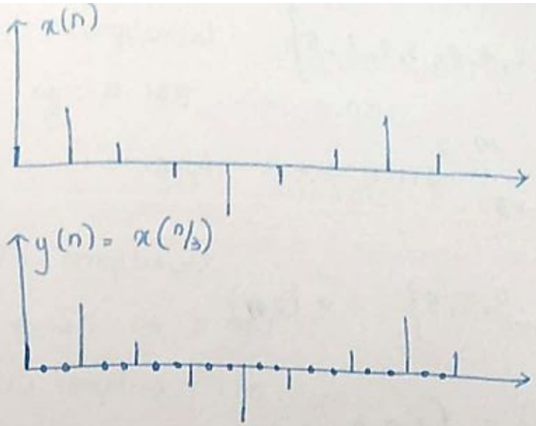


Up sampling (Interpolation)

The sampling rate of a discrete time signal can be increased by a factor L by placing $L-1$ equally spaced zeroes b/w each pair of samples. Mathematically, up sampling is represented by $y(n) = x(n/L)$

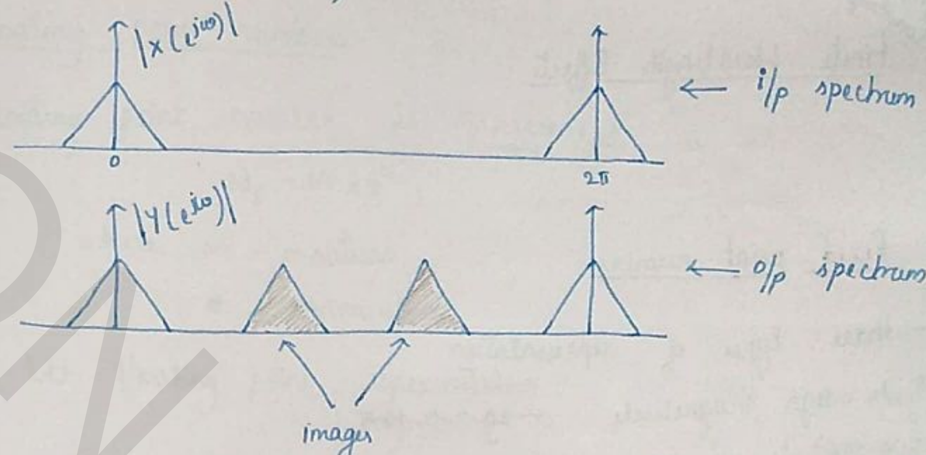


Block diagram

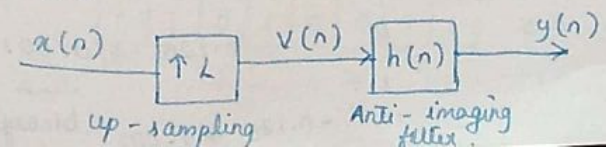


Spectrum

$$Y(e^{j\omega}) = X(e^{j\omega L})$$



The above spectrum is sketched for $L = 3$. Here we can see that the o/p spectrum have images in it. This is due to inserting $L-1$ zeroes ~~with~~ b/w successive samples of the i/p spectrum. Inorder to avoid imaging we use an anti-imaging filter after up sampling.



Q. 1. $x(n) = \{1, -1, 2, 4, 0, 3, 2, 1, 5\}$

(a) Down sampling $M=2$

(b) up sampling $L=3$

soln: (a) $y(n) = \{1, 2, 0, 2, 5\} = x(2n)$

(b) $y(n) = x(n/3) = \{1,0,0,-1,0,0,2,0,0,4,0,0,0,0,0,3,0,0,2,0,0,1,0,0,5\}$

11/11/2019 Monday

Finite Wordlength Effects

(in phone)

Fixed point numbers

- three types of representation
  1. sign magnitude ~~eg. 0.125~~
  2. 1's complement
  3. 2's complement

1. sign magnitude

eg: 0.125

$0.125 \times 2 \rightarrow 0.25$ 0

$0.25 \times 2 \rightarrow 0.5$ 0

$0.5 \times 2 \rightarrow 1$ 1

$0.125 \rightarrow 0.001$ binary

$-0.125 \rightarrow 1.001$

2. 1's complement.

eg: $0.125 \rightarrow 0.001$

1's compl. $\rightarrow$ 1.110

3. 2's complement

$0.125 \rightarrow 0.001$

1's compl $\rightarrow 1.110$

$\quad\quad\quad 1+$

2's compl $\rightarrow \overline{1.111}$

Floating point numbers

Floating point number is represented as

$$N_f = M \times 2^E$$

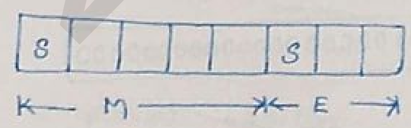
where, M – mantissa

E – exponent

* 8-bit floating point representation:

$1000 = 0.1 \times 10^4$

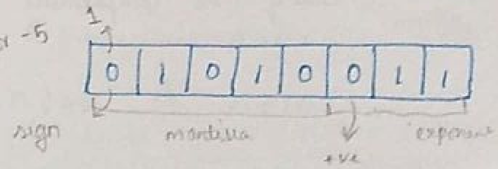
$1 = 1000 \times 10^{-3}$



→ floating point representation of +5

$+5 \rightarrow 101 = 0.101 \times 2^3 = 0.101 \times 2^{11}$ ($3 \rightarrow 11$)

for -5 → 1



→ +6 → 110 = 0.110×2^{3} = 0.110×2^{11}

0	1	1	0	0	0	1	1

→ 0.125 → 0.001 = 0.1×2^{-2} = 0.1×2^{-10} (binary of 2)

0	1	0	0	0	1	1	0

* 32-bit floating point representation

$N = (-1)^{S} \times 2^{E-127} \times M$

S	E	M
0	1 – 8	9 – 31

$A = E - 127$

$E = 131$

→ +25 → 11001 = $1.\underset{M}{1001} \times 2^{4}$ = $1.1001 \times 2^{131-127}$

0	10000011	10010000000000000000000

∵ +ve, 10000011 = 131

12/11/19 Tuesday

Fixed point arithmetics

Q.1. Add +0.375 & -0.625 using 1's complement addition.

0.375 → 0.011

0.625 → 0.101

-0.625 $\xrightarrow{1's}$ 1.010

```
  0.011
  1.010 +
  -----
  1.101
```

1.101 $\xrightarrow{1's}$ 1.010 = -0.010

$= 0 \times 2^{-1} + 1 \times 2^{-2} + 0 \times 2^{-3}$

= -0.25

2. +0.625 & -0.375

0.625 → 0.101

0.375 → 0.011

-0.375 $\xrightarrow{1's}$ 1.100

```
   0.101
   1.100 +
  ------
 (1)0.001
       1 +
  ------
   0.010
```

→ +0.25

Q2. 2's complement addition.

1. +0.375 & -0.625

0.375 → 0.011

0.625 → 0.101

-0.625 $\xrightarrow{1's}$ 1.010

```
  1.010
      1 +
  -----
  1.011
```

-0.625 $\xrightarrow{2's}$ 1.011

$0.011 + 1.011 = 1.110$

$1.110 \xrightarrow{1's} 01.001$

$01.001 + 1 = 01.010 \rightarrow -0.25$

2. $+0.625$ & -0.375

$0.625 \longrightarrow 0.101$

$0.375 \longrightarrow 0.011$

$-0.375 \xrightarrow{1's} 1.100$

$-0.375 \xrightarrow{2's} 1.100 + 1 = 1.101$

$0.101 + 01.101 = 10.010$ $\quad \times\boxed{1}0.010$

$= 0.010 = {}^{+}0.25$

Floating_point addition

Q.1. $5 + 0.25$

$5 \rightarrow 101 \rightarrow 0.101 \times 2^3 \longrightarrow 0.10100 \times 2^3$

$0.25 \rightarrow 0.010 \rightarrow 0.10 \times 2^{-1} \longrightarrow 0.00001 \times 2^3$

$0.10100 \times 2^3 + 0.00001 \times 2^3 = 0.10101 \times 2^3 \Rightarrow 0.10101 \times 2^{011}$

Floating_point multiplication

Q. 5×0.25

$5 \rightarrow 101 \rightarrow 0.101 \times 2^3 \rightarrow 0.10100 \times 2^3$

$0.25 \rightarrow 0.010 \rightarrow 0.10 \times 2^{-1} \longrightarrow 0.10000 \times 2^{-1}$

$= (0.10100 \times 0.10000) \times 2^2$

$= 0.0101000000 \times 2^2$

$= 0.10100 \times 2^{001}$

* Quantisation of input data (in ADC)

Variance of error signal.

Let us assume that $x(n)$ is an unquantised sample of a signal. $x_q(n)$ - quantised sample of signal. Then the quantisation error is given by:

$$e(n) = x_q(n) - x(n).$$

Let the range of an analog signal to be quantised be R. Then the quantisation step-size 'q' is given by:

$$q = \frac{R}{2^b}$$

We assume that all the errors are equiprobable. Also we treat the error as a random variable. The quantisation error for rounding will be in the range $-q/2$ to $+q/2$.

For a uniformly distributed random variable in the interval (x_1, x_2) the mean value & variance is given by:

$$\text{Mean, } E(x) = \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} x \cdot dx$$

$$\text{Variance, } \sigma^2 = E(x^2) - E^2(x)$$

$$E(\text{error}) = E(e) = \frac{1}{q/2 - (-q/2)} \int_{-q/2}^{q/2} e \, de$$

$$= \frac{1}{q} \left[\frac{e^2}{2}\right]_{-q/2}^{q/2} = \underline{0}$$

Variance, $\sigma_e^2 = E(e^2) - [E^2(e)]^2$

$$= E(e^2) \qquad [\because E(e) = 0]$$

$$E(e^2) = \frac{1}{\frac{q}{2} - (\frac{-q}{2})} \int_{-q/2}^{q/2} e^2 \, de$$

$$= \frac{1}{q} \left(\frac{e^3}{3}\right)_{-q/2}^{q/2} = \frac{1}{q}\left[\frac{q^3}{8 \times 3} + \frac{q^3}{8 \times 3}\right]$$

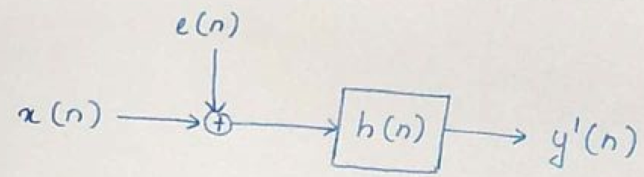
$$= \frac{2q^2}{24} = \frac{q^2}{12}$$

$$\sigma_e^2 = \frac{q^2}{12}$$

$\because q = \frac{R}{2^b}$,

$$\boxed{\sigma_e^2 = \frac{R^2}{12} \cdot 2^{-2b}}$$

ADC quantisation noise



$$y'(n) = x_q(n) * h(n)$$

$$= [x(n) + e(n)] * h(n)$$

$$= [x(n) * h(n)] + [e(n) * h(n)]$$

The steady state o/p noise power due to quantisation error is given by:

$$\sigma_{eo}^2 = \sigma_e^2 \sum_{n=0}^{\infty} h^2(n).$$

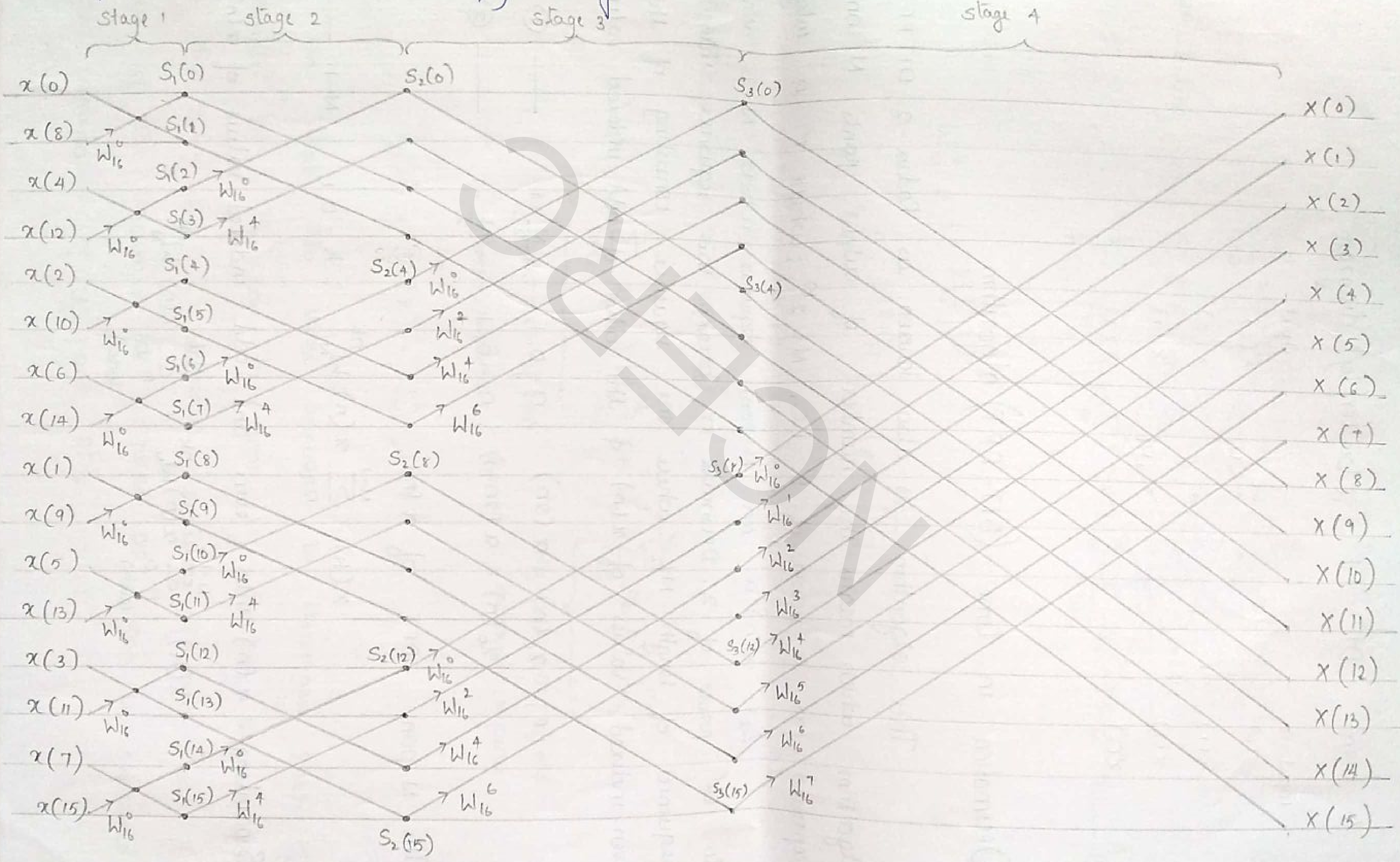
Using Parseval's theorem.

$$\sigma_{eo}^2 = \sigma_e^2 \sum_{n=0}^{\infty} h^2(n) = \sigma_e^2 \frac{1}{2\pi j} \oint_c H(z)\, H(z^{-1})\, z^{-1}\, dz.$$

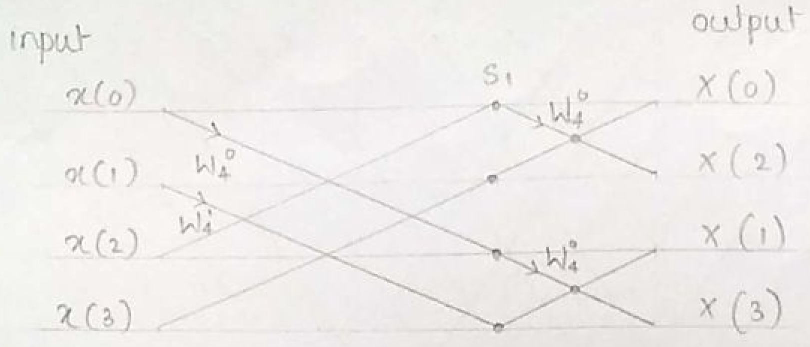
Using Residue theorem

$$\sigma_{eo}^2 = \sigma_e^2 \sum_{i=1}^{N} Res\left[H(z)\,H(z^{-1})\,z^{-1}\right]_{z=P_i}$$

16 - point DIT - FFT butterfly diagram



4 - point DIF - FFT butterfly diagram



Decimation - in - time (DIT - FFT) algorithm

This algorithm is also known as Radix - 2 DIT FFT algorithm which means the number of output points N can be expressed as a power of 2 i.e., $N = 2^M$; where M is integer.

Let $x(n)$ is an N-point sequence, where N is assumed to be power of 2. Decimate or break this sequence into two sequences of length N/2, where one sequence consisting of the even-indexed values of $x(n)$ & the other of odd-indexed values of $x(n)$

i.e, $x_e(n) = x(2n)$; $n = 0, 1, \ldots, \frac{N}{2} - 1$ —— (1)

$x_o(n) = x(2n+1)$; $n = 0, 1, \ldots, \frac{N}{2} - 1$ —— (2)

The N-point DFT of $x(n)$,

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk} \quad ; k = 0, 1, \ldots, N-1 \quad \text{—— (3)}$$

Separating $x(n)$ into even and odd indexed values of $x(n)$,

$$X(k) = \sum_{\substack{n=0 \\ (\text{even})}}^{N-1} x(n) W_N^{nk} + \sum_{\substack{n=0 \\ (\text{odd})}}^{N-1} x(n) . W_N^{nk}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(2n) W_N^{2nk} + \sum_{n=0}^{\frac{N}{2}-1} x(2n+1) W_N^{(2n+1)k}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(2n) W_N^{2nk} + W_N^{k} \sum_{n=0}^{\frac{N}{2}-1} x(2n+1) W_N^{2nk} \quad \text{--- (4)}$$

Substituting equ. (1) & (2) in (4),

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x_e(n) W_N^{2nk} + W_N^{k} \sum_{n=0}^{\frac{N}{2}-1} x_o(n) W_N^{2nk} \quad \text{--- (5)}$$

We have,

$$W_N^2 = \left(e^{-j\frac{2\pi}{N}}\right)^2$$

$$= \left(e^{-j 2\pi \times \frac{2}{N}}\right) = \left(e^{-j\frac{2\pi}{N/2}}\right)$$

$$W_N^2 = W_{N/2}$$

Substituting W_N^2 in equ. (5),

$$X(k) = \underbrace{\sum_{n=0}^{\frac{N}{2}-1} x_e(n) W_{N/2}^{nk}}_{N/2\text{-point DFT of even indexed sequence}} + W_N^{k} \underbrace{\sum_{n=0}^{\frac{N}{2}-1} x_o(n) W_{N/2}^{nk}}_{N/2\text{-point DFT of odd indexed sequence}}$$

$$X(k) = X_e(k) + W_N^{k} X_o(k)$$

Now we apply the same approach to decompose each of $\frac{N}{2}$ sample DFT. This can be done by dividing the sequence $x_e(n)$ and $x_o(n)$ into 2 sequences consisting of even & odd members of the sequences. The $\frac{N}{2}$ point DFTs can be expressed as a combination of $\frac{N}{4}$ point DFTs i.e., $X_e(k) = X_{ee}(k) + W_N^{2k} X_{eo}(k)$; $0 \le k \le \frac{N}{2} - 1$

$= X_{ee}\left(k - \frac{N}{4}\right) - W_N^{2\left(k - \frac{N}{4}\right)} X_{eo}\left(k - \frac{N}{4}\right)$; $\frac{N}{4} \le k \le \frac{N}{2} - 1$

where $X_{ee}(k)$ is $\frac{N}{4}$ point DFT of the even members of $x_e(n)$ and $X_{eo}(k)$ is $\frac{N}{4}$ point of DFT of the odd members of $x_e(n)$

In the same way,

$X_o(k) = X_{oe}(k) + W_N^{2k} X_{oo}(k)$; $0 \le k \le \frac{N}{2} - 1$

$= X_{oe}\left(k - \frac{N}{4}\right) - W_N^{2(k - N/4)} X_{oo}\left(k - \frac{N}{4}\right)$; $\frac{N}{4} \le k \le \frac{N}{2} - 1$

where $X_{oe}(k)$ is $\frac{N}{4}$ point DFT of the even members of $x_o(n)$ and $X_{oo}(k)$ is $\frac{N}{4}$ point of DFT of the odd members of $x_o(n)$

Decimation - in - Frequency algorithm

DIT algorithm is based on the decomposition of the DFT computation by forming smaller and smaller subsequences of the sequence $x(n)$. In DIF algorithm the output sequence $X(k)$ is divided into smaller & smaller subsequences. In this algorithm the input sequence $x(n)$ is partitioned into two sequences each of length $\frac{N}{2}$ samples. The first sequence $x_1(n)$ consists of first $\frac{N}{2}$ samples of $x(n)$ and the second sequence $x_2(n)$ consists of the last $\frac{N}{2}$ samples of $x(n)$.

i.e., $x_1(n) = x(n)$; $n = 0, 1, 2, \ldots, \frac{N}{2} - 1$ —— ①

$x_2(n) = x\left(n + \frac{N}{2}\right)$; $n = 0, 1, 2, \ldots, \frac{N}{2} - 1$ —— ②

If $N=8$ the first sequence $x_1(n)$ has values for $0 \le n \le 3$ & $x_2(n)$ has values for $4 \le n \le 7$. The N-point DFT of $x(n)$,

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{nk} + \sum_{n=\frac{N}{2}}^{N-1} x(n) W_N^{nk}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x_1(n) W_N^{nk} + \sum_{n=0}^{\frac{N}{2}-1} x_2(n) W_N^{(n+\frac{N}{2})k}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x_1(n) W_N^{nk} + W_N^{\frac{Nk}{2}} \sum_{n=0}^{\frac{N}{2}-1} x_2(n) W_N^{nk}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x_1(n) W_N^{nk} + e^{-j\pi k} \sum_{n=0}^{\frac{N}{2}-1} x_2(n) W_N^{nk}$$

When k is even, $e^{-j\pi k} = 1$.

$$X(2k) = \sum_{n=0}^{\frac{N}{2}-1} [x_1(n) + x_2(n)] W_N^{2nk}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} [x_1(n) + x_2(n)] W_{N/2}^{nk}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} f(n) W_{N/2}^{nk} \qquad \text{--- (3)}$$

where, $f(n) = [x_1(n) + x_2(n)]$

Equ. (3) is the $\frac{N}{2}$ point DFT of the $\frac{N}{2}$ point sequence $f(n)$ obtained by adding the first half & the last half of the input sequence.

When k is odd, $e^{-j\pi k} = -1$

$$X(2k+1) = \sum_{n=0}^{\frac{N}{2}-1} [x_1(n) - x_2(n)] W_N^{(2k+1)n}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} [x_1(n) - x_2(n)] W_N^{n} W_{N/2}^{nk}$$

$$X(2k+1) = \sum_{n=0}^{\frac{N}{2}-1} g(n)\, W_{N/2}^{nk} \quad \text{——} \quad (4)$$

where, $g(n) = [x_1(n) - x_2(n)]\, W_N^n$

Equ. (4) is the $\frac{N}{2}$ point DFT of the sequence $g(n)$ obtained by subtracting the second half of the input sequence from the first half and then multiplying the resulting sequence with W_N^n.